

# LOCALIZATION TECHNIQUES IN CIRCLE-EQUIVARIANT KASPAROV THEORY

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ABSTRACT. Let  $\mathbb{T}$  be the circle and  $A$  be a  $\mathbb{T}$ - $C^*$ -algebra. Then the  $\mathbb{T}$ -equivariant K-theory  $K_*^{\mathbb{T}}(A)$  is a module over the representation ring  $\text{Rep}(\mathbb{T})$  of the circle. The latter is a Laurent polynomial ring. Using the support of the module as an invariant, and techniques of Atiyah, Bott and Segal, we deduce that there are examples of  $\mathbb{T}$ - $C^*$ -algebras  $A$  such that  $A$  and  $A \rtimes \mathbb{T}$  are in the bootstrap category, but  $A$  is not  $\text{KK}^{\mathbb{T}}$ -equivalent to any commutative  $\mathbb{T}$ - $C^*$ -algebra. This is in contrast to the non-equivariant situation, in which any  $C^*$ -algebra in the bootstrap category is  $\text{KK}$ -equivalent to a commutative one. Our examples arise from dynamics, and include Cuntz-Krieger algebras with their usual circle actions. Using similar techniques, we also prove an equivariant version of the Lefschetz fixed-point formula. This is a special case of a result with Ralf Meyer that applies to general compact connected groups. The Lefschetz theorem equates the module trace of the module map of  $K_*^{\mathbb{T}}(X)$  induced by a morphism in  $\text{KK}^{\mathbb{T}}(C(X), C(X))$ , with an appropriate Kasparov product. When the morphism is the class of an equivariant correspondence, then the Kasparov product is the  $\mathbb{T}$ -equivariant index of the Dirac operator on a suitable ‘coincidence manifold’ of the correspondence. Finally, we prove several results related to localization and the Künneth and universal coefficient theorems, and give an essentially complete description of the  $\mathbb{T}$ -equivariant K-theory of compact spaces, by combining localization techniques of Atiyah and Segal and results of Paul Baum and Alain Connes for equivariant K-theory of finite group actions.

## 1. INTRODUCTION

This article has several purposes. The first is to show that many  $\mathbb{T}$ - $C^*$ -algebras are not  $\text{KK}^{\mathbb{T}}$ -equivalent to any commutative  $\mathbb{T}$ - $C^*$ -algebra, even though both they and their cross-products by  $\mathbb{T}$  are in the bootstrap category. These examples include the Cuntz-Krieger algebras  $O_A$  with their usual circle actions. The proof of this statement is very simple. We use a simple  $K_*^{\mathbb{T}}$ -theoretic obstruction to commutativity based on ideas of Atiyah, Bott and Segal.

One should note that this phenomenon is in sharp contrast to the non-equivariant situation, in which  $\text{KK}$  does not obstruct commutativity at all: in fact, as is well-known, every  $C^*$ -algebra in the bootstrap category is, up to  $\text{KK}$ -equivalence, commutative.

We will distinguish these classes by the support of the module  $K_*^{\mathbb{T}}(A)$  over the polynomial ring  $\text{Rep}(\mathbb{T})$ . Here  $K_*^{\mathbb{T}}(A)$  denotes equivariant K-theory with *complex coefficients*, i.e. is the integral integral K-theory tensored by  $\mathbb{C}$ . In particular  $\text{Rep}(\mathbb{T}) = \text{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[X, X^{-1}]$  is the ring of Laurent polynomials with complex coefficients.

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The support of  $K_*^{\mathbb{T}}(A)$  is always contained in the unit circle  $\mathbb{T} \subset \mathbb{C}^*$  if  $A$  is commutative (or is all of  $\mathbb{C}^*$ ). But, for a Cuntz-Krieger algebra  $O_A$ , it is the set of nonzero eigenvalues of the 0-1-valued matrix  $A$ . The conclusion is thus that  $O_A$  is not  $KK^{\mathbb{T}}$ -equivalent to any commutative  $\mathbb{T}$ - $C^*$ -algebra as soon as  $A$  has a nonzero eigenvalue not of modulus 1. In fact, therefore, the statement is somewhat stronger: the equivariant K-theory of  $O_A$  is not that of any commutative  $\mathbb{T}$ - $C^*$ -algebra, as a  $\mathbb{C}[X, X^{-1}]$ -module.

Study of equivariant K-theory groups  $K_G^*(X)$  as modules over  $\text{Rep}(G)$  began with a series of papers by Atiyah, Bott and Segal beginning in the 60's (see [1],[2],[3],[29].)

A common theme in these articles is to prove results about the case  $G = \mathbb{T}$ , and extend them using standard techniques to the case of connected groups  $G$  (we will restrict entirely to  $\mathbb{T}$  in this article.) The article [1] contains a treatment of equivariant K-theory (for torus actions) contains a lot of the essential ideas used by us here, except that we work in equivariant K-theory instead of equivariant cohomology. The other most important sources for equivariant K-theory are the articles [29] of Segal, and the second paper in the series on the Index Theorem [2]. As we wish to reach a wider readership than only those who are familiar with these articles, we have explained localization rather carefully in this article.

The work of Atiyah and Segal, shows that in the commutative case, the effect of localization of  $K_{\mathbb{T}}^*(X)$  can be described directly in terms of the action: it corresponds in a precise way to replacing  $X$  by fixed-point sets of elements of  $\mathbb{T}$ . The same pattern of argument shows that the support of  $K_{\mathbb{T}}^*(X)$  is always a finite set of roots of unity inside  $\mathbb{T} \subset \mathbb{C}^*$  – or is all of  $\mathbb{C}^*$ . By contrast, Cuntz-Krieger algebras, as noted above, have rather arbitrary algebraic integers as spectral points.

As well as making this observation, we aim in this article at strengthened versions of results on  $\mathbb{T}$ -equivariant K-theory of compact smooth manifolds due to Atiyah *et al.* For example, we prove that after a suitable localization,  $C(X)$  is  $KK^{\mathbb{T}}$ -equivalent to  $C(F)$  with  $F$  the stationary set. This is important for studying the Künneth and universal coefficient theorems for  $\mathbb{T}$ -spaces. This makes it possible to prove the *Lefschetz theorem* in  $KK^{\mathbb{T}}$  – see below. We also prove a number of detailed results about  $\mathbb{T}$ -equivariant K-theory that have a classical flavour, but which do not seem to have appeared anywhere.

For example, by combining the ideas of Atiyah and Segal with those of Baum and Connes, we give a more or less complete description of  $K_{\mathbb{T}}^*(X)$  for a compact smooth  $\mathbb{T}$ -manifold  $X$ . As far as we are aware, this is a new result. It is somewhat comparable to the K-theory computations in [10] for discrete proper group actions.

The smoothness hypothesis is convenient for several arguments; most importantly, it guarantees finite rank for  $K_{\mathbb{T}}^*(X)$  and finiteness of the support invariant for the torsion submodule, and allows us to compute certain boundary maps in terms of equivariant correspondences.

The third main purpose of this note is to give a simple proof of a  $\mathbb{T}$ -equivariant version of the Lefschetz fixed-point theorem. This is a special case of a result of the author and Ralf Meyer (see [16]). Our proof uses localization techniques, and is the motivation for much of the careful analysis in the middle part of this paper. We now describe this ‘Lefschetz theorem.’

Since  $\mathbb{C}[X, X^{-1}]$  is a principal ideal domain, any finitely generated  $\mathbb{C}[X, X^{-1}]$ -module  $M$  decomposes uniquely into a torsion module and a free module

$\cong \mathbb{C}[X, X^{-1}]^n$ . Any module map of  $M$  thus has a  $\mathbb{C}[X, X^{-1}]$ -valued *trace* by compressing it to the free part of  $M$ .

The equivariant analogue of the Lefschetz fixed-point theorem that we state here identifies the  $\mathbb{Z}/2$ -graded module trace of the  $\mathbb{C}[X, X^{-1}]$ -module map of  $K_{\mathbb{T}}^*(X)$  induced by the class of a  $\mathbb{T}$ -equivariant correspondence in  $KK_{\mathbb{T}}^*(C(X), C(X))$ , with a geometric invariant – the  $\mathbb{T}$ -index of the Dirac operator on a certain ‘coincidence manifold’ associated to the correspondence. This is closely related to a result of the author in the non-equivariant case (see [12]), and also to the Lefschetz *map* defined and studied in [15]. The equivariant Lefschetz theorem thus equates a geometric invariant of a correspondence, with a global, homological invariant.

The method of proof of such a theorem has already been explained in [12]; it involves Poincaré duality in an essential way, as well as the universal coefficient and Künneth theorems. The Lefschetz theorem, in short, uses all the essential properties of KK-theory – just as the classical version does for homology.

The equivariant version poses new problems, even for compact groups. The objective is to identify the geometric Lefschetz invariant — an element of  $\text{Rep}(G)$ , where  $G$  is a compact group – with a homological invariant like the  $\text{Rep}(G)$ -module trace of the endomorphism of  $K_G^*(X)$  coming from the initial data of a morphism  $f \in KK_G^*(C(X), C(X))$ . The module trace is, first of all, only defined in the usual way for projective  $\text{Rep}(G)$ -modules. There is no particular reason to expect  $K_G^*(X)$  to be a projective module in general (and it isn’t.) There is a method of overcoming this using the Hattori-Stallings trace, at least for groups  $G$  satisfying the Hodgkin condition (see [28]). But one needs to prove more about the geometric Lefschetz map to prove the theorem.

For  $G = \mathbb{T}$  there is no issue about defining module traces, as observed above. However, to prove the Lefschetz theorem one needs the Künneth theorem and universal coefficient theorems as well, and these are delicate in the equivariant situation. However, the work of Atiyah *et al* implies that after a suitable localization, we may as well replace  $X$  by the stationary set  $F \subset X$  of the action, which is a trivial  $\mathbb{T}$ -manifold. The Künneth and universal coefficient theorems and hence the Lefschetz theorem definitely hold for trivial  $\mathbb{T}$ -manifolds, and functoriality allows us to deduce the Lefschetz theorem for the original  $\mathbb{T}$ -manifold  $X$ .

The proof of the Lefschetz theorem in the general case will appear in the article [16]. It is significantly more elaborate, and uses projective resolutions instead of localization.

The other main source on equivariant KK-theory (aside from the fundamental [20]) is the article of Rosenberg and Schochet [28]. Their results are very extensive, but are integral, and localization is not pursued. Tensoring all K-theory groups by  $\mathbb{C}$  changes the flavour of equivariant K-theory drastically. (For instance the ring  $\mathbb{Z}[X, X^{-1}]$  is not a principal ideal domain, in contrast to  $\mathbb{C}[X, X^{-1}]$ .) Of course tensoring by  $\mathbb{C}$  results in loss of  $\mathbb{Z}$ -torsion information. But this seems to be more than compensated by the new algebro-geometric perspective that becomes available – and even leads to strong results about the integral case. For example, we observe that Cuntz-Krieger algebras are not  $KK^{\mathbb{T}} \otimes_{\mathbb{Z}} \mathbb{C}$ -equivalent to commutative ones, but this obviously implies they are not  $KK^{\mathbb{T}}$ -equivalent either.

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Higson for drawing my attention to the beautiful paper [1] of Atiyah and Bott on localization in equivariant cohomology.

In the following,  $\mathrm{KK}_*^{\mathbb{T}}(A, B)$  denotes the direct sum of  $\mathrm{KK}_0^{\mathbb{T}}(A, B)$  and  $\mathrm{KK}_1^{\mathbb{T}}(A, B)$ . Normally, we just write  $\mathrm{KK}^{\mathbb{T}}(A, B)$  for  $\mathrm{KK}_0^{\mathbb{T}}(A, B)$ .

## 2. THE $\mathbb{T}$ -SPECTRUM OF SPACES

In the following, the reader should consider all  $\mathbb{T}$ -equivariant K-theory groups, *e.g.*  $K_{\mathbb{T}}^*(X)$  for a  $\mathbb{T}$ -space  $X$ , or  $K_{\mathbb{T}}^{\mathbb{T}}(A)$  for a  $\mathbb{T}$ -C\*-algebra  $A$ , as tensored by the complex numbers. Similarly, the symbol  $\mathrm{Rep}(\mathbb{T})$  shall mean for us the usual representation ring of the circle, tensored with the complex numbers, or, more conveniently for us, the ring  $\mathbb{C}[X, X^{-1}]$  of Laurent polynomials in one variable, and complex coefficients. The isomorphism  $\mathrm{Rep}(\mathbb{T}) \rightarrow \mathbb{C}[X, X^{-1}]$  is the character map.

This note makes crucial use of the fact that for any  $\mathbb{T}$ -C\*-algebra  $A$ , the  $\mathbb{T}$ -equivariant K-theory  $K_{\mathbb{T}}^{\mathbb{T}}(A)$  is a module over  $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$ . For unital, commutative  $\mathbb{T}$ -C\*-algebras this is rather clear, since in this case  $K_{\mathbb{T}}^{\mathbb{T}}(A)$  is a ring and the unital inclusion  $\mathbb{C} \rightarrow A$  maps  $\mathrm{Rep}(\mathbb{T})$  to a subring of  $K_{\mathbb{T}}^{\mathbb{T}}(A)$ . This induces the module structure. It is not hard to convince oneself that if even if  $A$  is not unital, and hence no ring embedding exists, the module structure still makes sense.

In the general case, we may point to the *external product* in equivariant Kasparov theory as a formal definition of the module structure: to translate to Kasparov language,  $K_{\mathbb{T}}^{\mathbb{T}}(A) = \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A)$  and  $\mathrm{Rep}(\mathbb{T}) = \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C})$  (tensored by the complex numbers.) So Kasparov external product gives grading-preserving maps

$$\begin{aligned} \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A) \times \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) &\rightarrow \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A) \\ \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \times \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, A) &\rightarrow \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A) \end{aligned}$$

These maps agree: external product is commutative.

More generally,  $\mathrm{KK}_*^{\mathbb{T}}(A, B)$  is a graded  $\mathrm{Rep}(\mathbb{T})$ -module for any  $A, B$ .

For commutative  $A$ , *i.e.* for  $\mathbb{T}$ -spaces, the module structure of  $K_{\mathbb{T}}^*(X)$  over  $\mathrm{Rep}(\mathbb{T})$  has been quite extensively studied by Atiyah and Segal in [1] and [2], and also by Atiyah and Bott in the context of equivariant cohomology in [1].

The following definition applies to arbitrary  $\mathbb{C}[X, X^{-1}]$ -modules, and indeed, to modules over more general polynomial rings.

**Definition 2.1.** Let  $M$  be a module over the ring  $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$ . Its *annihilator*  $\mathrm{ann}(M)$  is the ideal  $\{f \in \mathbb{C}[X, X^{-1}] \mid fM = 0\}$ . Its *support* is defined by

$$\mathrm{supp}(M) := \bigcap_{f \in \mathrm{ann}(M)} Z_f$$

where  $Z_f \subset \mathbb{C}^*$  is the zero set of  $f$ .

The support of a module measures torsion: a point  $z$  is not in the support of  $M$  if and only if there is a polynomial  $f$  such that  $f(z) \neq 0$  but  $fM = 0$ . A free module has no torsion: the support of a free module like  $\mathbb{C}[X, X^{-1}]$  itself, is  $\mathbb{C}^*$ . Furthermore, under embeddings  $M_1 \rightarrow M_2$  of  $\mathbb{C}[X, X^{-1}]$ -modules, supports can only increase as  $\mathrm{ann}(M_2) \subset \mathrm{ann}(M_1)$  in this situation, which implies  $\mathrm{supp}(M_1) \subset \mathrm{supp}(M_2)$ . In particular,  $\mathrm{supp}(M) = \mathbb{C}^*$  as soon as  $M$  contains a free submodule.

$\mathbb{C}[X, X^{-1}]$  is a principal ideal domain, *i.e.* any ideal is generated by a single polynomial  $f$ . This polynomial is unique up to multiplication by an invertible in  $\mathbb{C}[X, X^{-1}]$ , *i.e.*  $f$  can be replaced by  $fX^n$  for any integer  $n$ , and in particular  $f$

may always be taken to be a polynomial. It is also elementary that any finitely generated module over a principal ideal domain is a direct sum of a free module and a torsion module. The torsion sub-module is by definition  $\{m \in M \mid fm = 0 \text{ for some } f \neq 0 \text{ in } \mathbb{C}[X, X^{-1}]\}$ .

A finitely generated torsion module has a nonzero annihilator ideal because the annihilator ideal is the intersection of the annihilator ideals of the generators, this is an intersection of finitely many nonzero ideals and hence is nonzero (*c.f.* Lemma 2.6 below). If the annihilator of the torsion module is generated by  $f$ , then the support of the torsion module is the zero set  $Z_f$  of  $f$  in  $\mathbb{C}^*$ .

If a module has finite dimension as a vector space over  $\mathbb{C}$  then the free summand must of course be zero; it is a finitely generated torsion module and the above discussion applies.

More generally, if  $M$  is any torsion module, of finite dimension over  $\mathbb{C}$  or not, and if the annihilator ideal of  $M$  is nonzero, then the same result holds: the support is the zero set of the generator  $f$  of the annihilator ideal. In this case, ring multiplication by  $X \in \mathbb{C}[X, X^{-1}]$  on the module is an invertible, complex linear operator (we are just working in the category of vector space maps. Invertibility means in this category.) Then *the support is exactly the set of eigenvalues of  $X$  and the generator  $f$  of the annihilator ideal is the minimal polynomial of  $X$* . Indeed, factor  $f(X) = (X - \lambda_1)^{k_1} \cdots (X - \lambda_n)^{k_n}$ . Each  $\lambda_i$  must be an eigenvalue of  $X$  since  $\prod_{j \neq i} (X - \lambda_j)^{k_j} (X - \lambda_i)^{k_i - 1}$  maps  $M$  into the kernel of  $X - \lambda_i$ . If the kernel of  $X - \lambda_i$  is zero, we would have a polynomial of smaller degree annihilating  $M$ , false. So the kernel is nonzero. Furthermore, as  $f(X) = 0$  on  $M$ ,  $0 = f(X)v = f(\lambda)v$  if  $v$  is any eigenvector of  $X$  with eigenvalue  $\lambda$ . Hence any eigenvalue of  $X$  is a root of  $f$ .

If the torsion module is finitely generated, the annihilator ideal is nonzero and so the above discussion applies and the support is a finite subset of  $\mathbb{C}^*$  and the set of eigenvalues of  $X$ .

*Remark 2.2.* Finite generation is guaranteed for the  $\mathbb{C}[X, X^{-1}]$ -module  $K_{\mathbb{T}}^*(X)$  whenever  $X$  is a smooth, compact manifold and  $\mathbb{T}$  acts smoothly (see [29]).

Let  $V = \bigoplus_1^\infty V_n$  be the sum of complex vector spaces  $V_n$ , let  $T_n$  be a linear automorphism of  $V_n$  with eigenvalues the group  $\Omega_n \subset \mathbb{T}$  of complex  $n$ th roots of unity and let  $X$  act on the sum with  $X$  acting by  $T_n$  on  $V_n$ . Then  $V$  is torsion,  $X$  has eigenvalues  $\bigcup_n \Omega_n$ , but the annihilator ideal of the module is zero and the support is  $\mathbb{C}^*$ , so the support of  $V$  differs from the set of eigenvalues of  $X$  in this case, which it must, of course, since the support is always Zariski closed. See Example 2.8 for more discussion.

The classification of  $\mathbb{C}[X, X^{-1}]$ -modules which are finite-dimensional over  $\mathbb{C}$  is exactly the problem of classifying invertible matrices up to similarity. In the language of matrices, a complete invariant is the Jordan canonical form. For now we are mainly just focused on the spectrum – *i.e.* the set of eigenvalues of  $X$ .

**Definition 2.3.** Let  $A$  be a  $\mathbb{T}$ - $C^*$ -algebra. The  $\mathbb{T}$ -spectrum of  $A$  is defined to be the support of  $K_*^{\mathbb{T}}(A)$  as an  $\mathbb{C}[X, X^{-1}]$ -module.

In the commutative case, we refer to the  $\mathbb{T}$ -spectrum of the corresponding space.

The simplest example is the trivial  $\mathbb{T}$ -action on a point: of course then  $K_{\mathbb{T}}^*(\cdot) = \text{Rep}(\mathbb{T})$  has no annihilator and thus  $\mathbb{T}\text{-spec}(\cdot) = \mathbb{C}^*$ .

If  $A = C([0, \infty))$  with trivial  $\mathbb{T}$ -action, then  $K_*^{\mathbb{T}}(A) = 0$  and hence  $\mathbb{T}\text{-spec}(A) = \emptyset$  in this case.

For a *free* action, the support of  $K_{\mathbb{T}}^*(X)$  is  $\{1\} \in \mathbb{C}$ .

Although evaluation of Laurant polynomials at any  $z \in \mathbb{C}^*$  yields a  $\mathbb{C}[X, X^{-1}]$ -module  $M$  such that  $\text{supp}(M) = \{z\}$ , if this module is to arise from an equivariant K-theory module, then  $z$  must be an algebraic integer, at least if the module is finite dimensional over  $\mathbb{C}$ .

**Proposition 2.4.** *If  $K_*^{\mathbb{T}}(A)$  is finite-dimensional over  $\mathbb{C}$ , then the  $\mathbb{T}$ -spectrum of  $A$  is a finite set of algebraic integers in  $\mathbb{C}^*$ .*

*Proof.* The spectrum in this case is the spectrum of  $X$  acting on  $K_*^{\mathbb{T}}(A)$ . But  $X$  comes from an endomorphism of the underlying  $\mathbb{T}$ -equivariant K-theory with *integer coefficients* and therefore is represented in some basis for  $K_*^{\mathbb{T}}(A)$  by a matrix with integer coefficients, and  $\mathbb{T}\text{-spec}(A)$  is its set of eigenvalues, so they are algebraic integers.  $\square$

**Theorem 2.5.** *If  $A = C_0(X)$  is any commutative  $\mathbb{T}$ - $C^*$ -algebra, then either  $\mathbb{T}\text{-spec}(A) = \mathbb{C}^*$  or  $\mathbb{T}\text{-spec}(A) \subset \mathbb{T}$ . In the latter case, every point of the  $\mathbb{T}$ -spectrum is an  $n$ th complex root of unity for  $n$  the least common multiple of the orders of the isotropy groups for points in  $\mathbb{T} \backslash X$ . If  $X$  is compact, then  $\mathbb{T}\text{-spec}(X) = \mathbb{C}^*$  if and only if  $X$  has a stationary point.*

Start by assuming that  $X$  has a stationary point. Then there is a  $\mathbb{T}$ -map from the one-point  $\mathbb{T}$ -space to  $X$ ; it induces a module map  $K_{\mathbb{T}}^*(X) \rightarrow K_{\mathbb{T}}^*(\cdot) = \text{Rep}(\mathbb{T})$ . If  $X$  is *compact* this map is injective, because the map from  $X$  to a point is proper in this case and gives a splitting. Thus,  $\mathbb{T}\text{-spec}(X) = \mathbb{C}^*$  if  $X$  has a stationary point and is compact. This is rather common; for example, by the Hopf theorem any smooth  $\mathbb{T}$ -action on a smooth manifold of nonzero Euler characteristic has a stationary point. Hence having  $\mathbb{T}$ -spectrum  $\mathbb{C}^*$  is rather generic for compact  $\mathbb{T}$ -spaces.

If  $X$  is not compact, it may have a stationary point without the spectrum being  $\mathbb{C}^*$ ; for example  $[0, \infty)$  with trivial  $\mathbb{T}$ -action has empty spectrum but many stationary points. The other implication also requires compactness in view of Example 2.8.

For emphasis, we state the following.

**Lemma 2.6.** *If  $(I_{\lambda})_{\lambda \in \Lambda}$  is any infinite collection of (distinct) ideals in  $\mathbb{C}[X, X^{-1}]$ , then their intersection is zero. In general, the zero set of the intersection is the Zariski-closure of the union of the zero sets of the  $I_i$ .*

*Proof.* Indeed, every ideal in  $\mathbb{C}[X, X^{-1}]$  is of the form  $(f)$  for some polynomial  $f \in \mathbb{C}[X, X^{-1}]$ , and  $(f) \subset (g)$  if and only if  $g$  is a divisor of  $f$ . Hence there are only finitely many ideals containing a given one. The other statement is equally easy to check.  $\square$

For any collection  $(M_{\lambda})_{\lambda \in \Lambda}$  of nonzero  $\mathbb{C}[X, X^{-1}]$ -modules, the annihilator of the direct sum  $M := \bigoplus M_i$  is the intersection  $\bigcap_i \text{ann}(M_i)$  of the annihilators. Therefore, if there are infinitely many distinct  $\text{ann}(M_i)$ , then the annihilator of  $M$  is zero and the spectrum is  $\mathbb{C}^*$  in this case.

**Lemma 2.7.** *If  $X$  is a  $\mathbb{T}$ -space which is a disjoint union  $X = \bigsqcup_i X_i$  for a family of  $\mathbb{T}$ -spaces  $X_i$ . Then either the  $\mathbb{T}$ -spectrum of  $X$  is  $\mathbb{C}^*$  or the sets  $\mathbb{T}\text{-spec}(X_i)$  are all finite, there are only finitely many of them, and  $\mathbb{T}\text{-spec}(X)$  is their union.*

*Proof.*  $K_{\mathbb{T}}^*(X) = \bigoplus_{i \in \Lambda} K_{\mathbb{T}}^*(X_i)$  and the result follows from the preceding remarks.  $\square$

*Example 2.8.* Let  $\mathbb{T}$  act on  $X_n := \mathbb{T}$  with  $t \cdot s := t^n s$ . Let  $\Omega_n \subset \mathbb{T}$  denote the subgroup of  $n$ th complex root of unity. Then  $X_n \cong \mathbb{T}/\Omega_n$  with  $\mathbb{T}$  acting by translation on the quotient. Thus  $K_{\mathbb{T}}^*(X_n) \cong \text{Rep}(\Omega_n)$ , and the  $\mathbb{C}[X, X^{-1}] \cong \text{Rep}(T)$  module structure is by restriction of representations, *i.e.* by restrictions of polynomials to  $\Omega_n \subset \mathbb{C}^*$ . The support is  $\Omega_n$ , thus  $\mathbb{T}\text{-spec}(X_n) = \Omega_n$ .

Now let  $X = \mathbb{T} \times \mathbb{N}$  with  $\mathbb{T}$  acting as above in the  $n$ th copy of  $\mathbb{T}$ . By Lemma 2.7, the  $\mathbb{T}$ -spectrum of  $X$  is  $\mathbb{C}^*$ , although there is no stationary point.

The proof of Theorem 2.5 follows from the following two lemmas.

**Lemma 2.9.** *Let  $X$  be any  $\mathbb{T}$ -space and  $Y \subset X$  is a closed  $\mathbb{T}$ -invariant subspace of  $X$ . Then*

$$\mathbb{T}\text{-spec}(X) \subset \mathbb{T}\text{-spec}(Y) \cup \mathbb{T}\text{-spec}(X - Y).$$

*Proof.* Consider the 6-term exact sequence of  $\mathbb{T}$ -equivariant K-theory groups associated to the exact sequence

$$0 \rightarrow C_0(X - Y) \rightarrow C(X) \rightarrow C_0(Y) \rightarrow 0.$$

By the Five Lemma, if  $f \in \mathbb{C}[X, X^{-1}]$  annihilates  $K_{\mathbb{T}}^*(X - Y)$  and annihilates  $K_{\mathbb{T}}^*(Y)$ , then it annihilates  $K_{\mathbb{T}}^*(X)$ . Thus

$$\text{ann}(K_{\mathbb{T}}^*(X - Y)) \cap \text{ann}(K_{\mathbb{T}}^*(Y)) \subset \text{ann}(K_{\mathbb{T}}^*(X)).$$

Hence  $\mathbb{T}\text{-spec}(X) \subset \mathbb{T}\text{-spec}(X - Y) \cup \mathbb{T}\text{-spec}(Y)$  as claimed.  $\square$

**Lemma 2.10.** *If  $X := \mathbb{T} \times_H Y$  for some closed subgroup  $H \subset \mathbb{T}$  and some  $H$ -space  $Y$ , then  $\mathbb{T}\text{-spec}(X) \subset H$ . In particular, if  $H$  is a proper subgroup, then the  $\mathbb{T}$ -spectrum of  $X$  consists of a set of  $n$ th roots of unity, where  $n$  is the cardinality of  $H$ .*

*Proof.* The  $\text{Rep}(\mathbb{T})$ -module structure on  $K_{\mathbb{T}}^*(X) \cong K_H^*(Y)$  factors through the restriction map  $\text{Rep}(\mathbb{T}) \rightarrow \text{Rep}(H)$  and the  $\text{Rep}(H)$ -module structure on  $K_H^*(Y)$ . If  $f$  is a polynomial which vanishes on  $H \subset \mathbb{T}$  then it restricts to zero in  $\text{Rep}(H)$  and hence acts by zero on  $K_H^*(Y) \cong K_{\mathbb{T}}^*(X)$ . Hence the annihilator of  $K_{\mathbb{T}}^*(X)$  is contained in the zero set of  $f$ . This  $\mathbb{T}\text{-spec}(X) \subset H$  as claimed.  $\square$

*Remark 2.11.* We remind the reader of two easy and well-known facts about induced spaces.

- Induced spaces  $W = \mathbb{T} \times_H Y$  from a subgroup  $H \subset \mathbb{T}$  are characterised among  $\mathbb{T}$ -spaces as those admitting a  $\mathbb{T}$ -map  $\varphi: W \rightarrow \mathbb{T}/H$ . We can recover  $Y$  from  $\varphi$  as the fibre over the identity coset in  $\mathbb{T}/H$ .
- We often call induced spaces *slices*. Since we can always restrict a  $\mathbb{T}$ -map to a  $\mathbb{T}$ -invariant subspace, any  $\mathbb{T}$ -invariant subspace of a slice is a slice too.
- A theorem of Palais (see [23]) asserts that  $\mathbb{T}$ -space can be covered by *open* slices using stabilizer subgroups of the action. That is, if  $X$  is any  $\mathbb{T}$ -space and  $x \in X$ , then there exists an open subset  $U \subset X$  with  $x \in U$ , and a  $\mathbb{T}$ -map  $\varphi: U \rightarrow \mathbb{T}/H$  where  $H := \mathbb{T}_x$  is the stabilizer of  $x$ .

Note that if  $\varphi: U \rightarrow \mathbb{T}/H$  is a slice with  $H = \mathbb{T}_x$  for some  $x \in U$ , then  $\mathbb{T}_y \subset \mathbb{T}_x$  for any  $y \in U$ .

**Lemma 2.12.** *Let  $X$  be a (locally compact)  $\mathbb{T}$ -space.*

- *If  $X$  has no stationary points, then  $K_{\mathbb{T}}^*(X)$  is a torsion module and  $\mathbb{T}\text{-spec}(Y)$  is a finite subset of  $\mathbb{T}$  for every pre-compact  $\mathbb{T}$ -invariant subset  $Y \subset X$ . Furthermore,  $\mathbb{T}\text{-spec}(Y) \subset \bigcup_{y \in \overline{Y}} \mathbb{T}_y$ .*
- *If  $\mathbb{T}\text{-spec}(X)$  is finite,  $F \subset X$  is the stationary set, then  $K_{\mathbb{T}}^*(F) = 0$  and the  $\mathbb{T}$ -equivariant  $*$ -homomorphism  $C_0(X - F) \rightarrow C_0(X)$  determines an isomorphism  $K_{\mathbb{T}}^*(X - F) \cong K_{\mathbb{T}}^*(X)$  of  $\mathbb{C}[X, X^{-1}]$ -modules.*

*Remark 2.13.* The same arguments prove that the  $\mathbb{T}$ -equivariant  $*$ -homomorphism  $C_0(X - F) \rightarrow C_0(X)$  is invertible in  $\text{KK}^{\mathbb{T}}(C_0(X - F), C_0(X))$ .

The condition of having finite  $\mathbb{T}$ -spectrum thus implies that the stationary set  $F$  is *homologically trivial*: that is,  $K_{\mathbb{T}}^*(F) = 0$ . Compare the ray  $[0, \infty)$  with the trivial action.

*Proof.* For the first statement,  $K_{\mathbb{T}}^*(X)$  is the inductive limit of the  $K_{\mathbb{T}}^*(Y)$ , as  $Y \subset X$  ranges over the pre-compact  $\mathbb{T}$ -invariant subsets of  $X$ . Therefore, if we can prove that the annihilator ideal of  $K_{\mathbb{T}}^*(Y)$  is nonzero for every pre-compact  $\mathbb{T}$ -invariant subset  $Y \subset X$ , we will be done. This is equivalent to showing that  $\mathbb{T}\text{-spec}(Y)$  is finite for all such  $Y$ . If  $Y \subset X$  is precompact, with closure  $\overline{Y}$ , then we can cover  $\overline{Y}$  by finitely many  $\mathbb{T}$ -slices  $\varphi: U_i \rightarrow \mathbb{T}/H_i$  using stabilizer subgroups  $H_i$  of the action on  $\overline{Y}$ . This gives a finite cover of  $Y$  itself by open slices (as in Remark 2.11, intersecting a slice with a  $\mathbb{T}$ -invariant subset always results in a slice.) Furthermore, since the  $H_i$  are stabilizer groups of points in  $\overline{Y}$  and the action has no stationary points, all  $H_i$  are finite subgroups of  $\mathbb{T}$ .

Now prove the result by induction on the number of slices required to cover  $Y$ . It can be covered by a single slice, then it is itself a slice, and the result follows from Lemma 2.10. If the result is true for precompact subsets of  $X$  that can be covered by  $< n$  slices, and  $Y$  can be covered by  $n$  slices with domains  $U_1, \dots, U_n$  and subgroups  $H_i$ , then the closed  $\mathbb{T}$ -invariant subspace  $Y - U_n$  of  $Y$  can be covered by  $n - 1$  slices so by inductive hypothesis  $\mathbb{T}\text{-spec}(Y - U_n) \subset \bigsqcup_{x \in Y} \mathbb{T}_x \subset \mathbb{T}$  is finite. The result for  $Y$  now follows from Lemma 2.9.

For the second statement, consider the exact sequence of  $\mathbb{T}$ - $C^*$ -algebras

$$0 \rightarrow C_0(X - F) \rightarrow C_0(X) \rightarrow C_0(F) \rightarrow 0.$$

This induces an exact sequence of  $K_{\mathbb{T}}^*$ -groups. The restriction map  $K_{\mathbb{T}}^*(X) \rightarrow K_{\mathbb{T}}^*(F)$  must vanish, because we have assumed that  $X$  has finite spectrum, (*i.e.*  $K_{\mathbb{T}}^*(X)$  is torsion) whereas  $K_{\mathbb{T}}^*(F)$  is free. This implies that we have a pair of short exact sequences

$$0 \rightarrow K_{\mathbb{T}}^{*+1}(F) \rightarrow K_{\mathbb{T}}^*(X - F) \rightarrow K_{\mathbb{T}}^*(X) \rightarrow 0$$

for  $*$  = 0, 1. But  $X - F$  has no stationary points, so from the first part of this Lemma,  $K_{\mathbb{T}}^*(X - F)$  is torsion. Now a free  $\mathbb{C}[X, X^{-1}]$ -module  $K_{\mathbb{T}}^{*+1}(F)$  which injects into a torsion module  $K_{\mathbb{T}}^*(X - F)$  can only be the zero module. Hence  $K_{\mathbb{T}}^*(F) = 0$ ,  $*$  = 0, 1 and  $C_0(X - F) \rightarrow C_0(X)$  induces an isomorphism on  $K_{\mathbb{T}}^*$ -theory.  $\square$

*Proof. (Of Theorem 2.5.)* Assume that  $\mathbb{T}\text{-spec}(X) \neq \mathbb{C}^*$ .

Then since  $X$  has finite  $\mathbb{T}$ -spectrum,  $K_{\mathbb{T}}^*(X) \cong K_{\mathbb{T}}^*(X - F)$  as  $\mathbb{C}[X, X^{-1}]$ -modules, where  $F \subset X$  is the stationary set, by Lemma 2.12. In particular,  $\mathbb{T}\text{-spec}(X) = \mathbb{T}\text{-spec}(X - F)$  so by replacing  $X$  by  $X - F$  we may assume that  $X$  itself has no stationary points.



Suppose  $v \in K_{\mathbb{T}}^*(X)$ ,  $f \in \mathbb{C}[X, X^{-1}]$  minimal such that  $fv = 0$ . Since  $K_{\mathbb{T}}^*(X)$  is the inductive limit of the  $K_{\mathbb{T}}^*(Y)$  as  $Y \subset X$  ranges over the precompact  $\mathbb{T}$ -invariant subsets of  $X$ , there exists precompact  $Y$  and  $w \in K_{\mathbb{T}}^*(Y)$  mapping to  $v$ . By Lemma 2.12,  $K_{\mathbb{T}}^*(Y)$  has finite spectrum, *i.e.* has annihilator generated by some  $g \in \mathbb{C}[X, X^{-1}]$ . Since  $gw = 0$ ,  $gv = 0$  and since  $f$  is minimal,  $g$  contains  $f$  as a factor and so  $Z_f \subset Z_g$ . Since  $Z_g$  is contained in the unit circle, so is  $Z_f$ , and we are done.  $\square$

### 3. THE $\mathbb{T}$ -SPECTRA OF $C^*$ -ALGEBRAS

Let  $B$  be a  $C^*$ -algebra equipped with an automorphism  $\sigma$ . Then  $A := B \rtimes \mathbb{Z}$  is a  $\mathbb{T}$ - $C^*$ -algebra using the dual action

$$z(\sum_{n \in \mathbb{Z}} b_n[n]) := \sum_{n \in \mathbb{Z}} z^n b_n[n].$$

Hence it has a  $\mathbb{T}$ -spectrum. Recall that the equivariant K-theory of  $A$  is isomorphic to the K-theory  $K_*(A \rtimes \mathbb{T})$  of the cross-product. By Takai-Takesaki duality, this agrees with  $K_*(B)$ .

**Proposition 3.1.** *Let  $B$  be a  $C^*$ -algebra and  $\sigma \in \text{Aut}(B)$ . Endow the cross-product  $A := B \rtimes_{\sigma} \mathbb{Z}$  with the dual action of  $\mathbb{T} \cong \widehat{\mathbb{Z}}$ . The automorphism induces an invertible linear map  $\sigma_*: K_*(B) \rightarrow K_*(B)$  and hence a  $\mathbb{C}[X, X^{-1}]$  module structure on  $K_*(B)$ . This module is naturally isomorphic to  $K_{\mathbb{T}}^*(A)$ .*

*In particular, if  $K_*(B)$  is finite dimensional over  $\mathbb{C}$ , then*

$$\mathbb{T}\text{-spec}(B \rtimes \mathbb{Z}) = \text{Spec}(\sigma_*),$$

*with  $\text{Spec}(\sigma_*)$  the set of eigenvalues of the invertible linear map  $\sigma_* \in \text{End}_{\mathbb{C}}(K_*(B))$ .*

*Proof.* By Blackadar Proposition 11.8.3, the isomorphism

$$K_*(B) \cong K_*(B \rtimes \mathbb{Z} \rtimes \mathbb{T}) = K_*(A \rtimes \mathbb{T}) \cong K_{\mathbb{T}}^*(A)$$

of Takai-Takesaki duality and the Green-Julg theorem, intertwines the group homomorphism  $\sigma_*$  and the group homomorphism of scalar multiplication by  $X \in \mathbb{C}[X, X^{-1}] \cong \text{Rep}(\mathbb{T})$ .

Now by the same argument as in the commutative case, if  $K_{\mathbb{T}}^*(A)$  has finite dimension, then  $\mathbb{T}\text{-spec}(A)$  is the set of non-zero eigenvalues of the linear map  $X$  because it is the zero set of the minimal polynomial of the linear map  $X$ , which is its spectrum.  $\square$

*Remark 3.2.* Baaj-Skandalis duality (see [4]) is a functor  $\text{KK}^{\mathbb{T}} \rightarrow \text{KK}^{\mathbb{Z}}$  which on objects sends a  $\mathbb{T}$ - $C^*$ -algebra  $B$  to the  $\mathbb{Z}$ - $C^*$ -algebra  $B := A \rtimes \mathbb{T}$ , with the dual action and sends a  $\mathbb{T}$ -equivariant  $*$ -homomorphism  $A \rightarrow A'$  to the (obvious) induced map  $B := A \rtimes \mathbb{T} \rightarrow B' := A' \rtimes \mathbb{T}$ . Baaj and Skandalis extend this to a natural isomorphism

$$\text{KK}_{\mathbb{T}}^{\mathbb{T}}(A, A') \cong \text{KK}_{\mathbb{T}}^{\mathbb{Z}}(A \rtimes \mathbb{T}, A' \rtimes \mathbb{T})$$

of equivariant KK-groups. Note that this transformation maps an induced space  $X = \mathbb{T} \times_H Y$  for some  $H$ -space  $Y$  and a proper closed subgroup of  $\mathbb{T}$  to the  $\mathbb{Z}$ - $C^*$ -algebra

$$C(X) \rtimes \mathbb{T} \cong C_0(Y) \rtimes H$$

with an appropriate dual action. The important point is that this  $\mathbb{Z}$ - $C^*$ -algebra, or this  $\mathbb{Z}$ -action on a  $C^*$ -algebra, *factors through a periodic action*, i.e. factors through the homomorphism  $\mathbb{Z} \rightarrow H \cong \mathbb{Z}/n$  for some  $n$ , and a  $\mathbb{Z}/n$ -action.

Under the Baaj-Skandalis transformation, the  $\mathbb{T}$ -spectrum of a  $\mathbb{T}$ - $C^*$ -algebra  $A$  corresponds, as we have observed above, to the spectrum, in the usual sense, of the endomorphism of  $K_*(A)$  by the generator  $1 \in \mathbb{Z}$  of the  $\mathbb{Z}$ -action. Hence if the  $\mathbb{Z}$ -action is periodic, then the corresponding linear map has finite order, and hence its spectrum consists of roots of unity in the circle. This is, roughly, then, the counterpart of the situation in the first section, in the category  $\text{KK}^{\mathbb{Z}}$ .

For instance let  $A = \mathbb{C}$  with the trivial automorphism. Applying the proposition gives that the  $\mathbb{T}$ -spectrum of  $C^*(\mathbb{Z})$  with its dual action of  $\mathbb{T}$  is the single point  $\{1\} \subset \mathbb{C}^*$ .

*Example 3.3.* The  $\mathbb{T}$ -spectrum of the irrational rotation algebra  $A_\theta := C(\mathbb{T}) \rtimes_{R_\theta} \mathbb{Z}$  with the dual action of  $\mathbb{T}$  is also  $\{1\}$  because  $\sigma_*$  is the identity map on  $K^*(\mathbb{T})$ .

*Example 3.4.* Let  $A$  be an integer matrix with entries either 0 or 1. Then the  $\mathbb{T}$ -spectrum of the associated Cuntz-Krieger algebra  $O_A$  is the set of nonzero eigenvalues of  $A$  because  $O_A \cong F_A \rtimes \mathbb{Z}$  where  $F_A$  is an appropriate AF-algebra, and  $\cong$  means Morita equivalence. Assuming for simplicity that  $A$  is actually invertible, it is well-known and easily checked from the Bratteli diagram, that the K-theory of  $F_A$  is  $\cong \mathbb{C}^n$ , and the action of  $\mathbb{Z}$  on it is by the matrix  $A$ .

**Corollary 3.5.** *The Cuntz-Krieger algebra  $O_A$  is not  $\text{KK}^{\mathbb{T}}$ -equivalent to any unital commutative  $\mathbb{T}$ - $C^*$ -algebra as soon as the integer matrix  $A$  has some eigenvalue of modulus  $\neq 1$ .*

This happens for instance if  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ .

For the benefit of the reader (the result is well-known) we prove the following.

**Lemma 3.6.** *Both  $O_A$  and  $O_A \rtimes \mathbb{T} \cong F_A$  are in the bootstrap category  $\mathcal{N}$ .*

*Proof.*  $F_A$  is an AF algebra so is in  $\mathcal{N}$ . The Baum-Connes conjecture for  $\mathbb{Z}$  is the statement that  $C_0(\mathbb{R})$  with the  $\mathbb{Z}$ -action by translation is  $\text{KK}_1^{\mathbb{Z}}$ -equivalent to  $\mathbb{C}$ . It follows from this that  $O_A = F_A \rtimes \mathbb{Z}$  is  $\text{KK}$ -equivalent to  $C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z}$ . There is an exact sequence

$$0 \rightarrow S \otimes F_A \otimes \mathbb{K} \rightarrow C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z} \rightarrow F_A \otimes \mathbb{K} \rightarrow 0$$

of  $C^*$ -algebras, obtained by evaluating functions on  $\mathbb{R}$  at the integer points  $\mathbb{Z} \subset \mathbb{R}$ , a closed and  $\mathbb{Z}$ -invariant subset, and using  $C_0(\mathbb{Z}) \rtimes \mathbb{Z} \cong \mathbb{K}$ . Since  $\mathbb{K}$  is  $\text{KK}$ -equivalent to  $\mathbb{C}$  both ends are in the bootstrap category. Hence  $C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z}$  is also.  $\square$

*Remark 3.7.* We have actually proved something stronger than Corollary 3.5, for we have shown that the  $\mathbb{T}$ -equivariant K-theory of  $O_A$  is not isomorphic in the category of  $\mathbb{C}[X, X^{-1}]$ -modules to the  $\mathbb{T}$ -equivariant K-theory of any locally compact Hausdorff  $\mathbb{T}$ -space.

We close this section with some further remarks on  $\mathbb{T}$ -equivariant K-theory of Cuntz-Krieger algebras, to point out the connection to zeta functions.

An invariant of the  $\mathbb{Z}/2$ -graded  $\mathbb{C}[X, X^{-1}]$ -module  $K_*^{\mathbb{T}}(A)$  – assuming it finite dimensional over  $\mathbb{C}$  – is the rational function

$$(3.1) \quad \text{char}_A(t) := \frac{\det(1 - tX_+)}{\det(1 - tX_-)}$$

where  $X_{\pm}$  denotes the action of the generator  $X$  on  $K_*^{\mathbb{T}}(A)$ .

If  $A$  and  $B$  are  $\text{KK}^{\mathbb{T}}$ -equivalent, they have the same rational function (3.1).

The following elementary result about linear transformations can be found in the appendices to Hartshorne's book [19]:

$$(3.2) \quad \text{char}_A(t) = \exp\left(\sum_{n=1}^{\infty} \text{trace}_s(X^n) \frac{t^n}{n}\right)$$

holds, where  $\text{trace}_s$  is the graded trace, the difference of the traces of  $X$  acting on  $K_1^{\mathbb{T}}(A)$  and  $K_0^{\mathbb{T}}(A)$ .

We now specialize to the following situation: let  $\phi_T: \mathbb{T}^n \rightarrow \mathbb{T}^n$  be a linear automorphism, *i.e.*  $\phi$  is translation by an element  $T \in \text{GL}_n(\mathbb{Z})$ . We assume that  $T$  is self-adjoint, so it is diagonalizable over  $\mathbb{C}$  with real, nonzero eigenvalues. We can form the cross-product  $A := C(\mathbb{T}^n) \rtimes_{\phi_T} \mathbb{Z}$ , which is a  $\mathbb{T}$ - $C^*$ -algebra. By the Lefschetz fixed-point theorem,

$$\text{trace}_s((\phi_T^*)^n) = (-1)^k P_n(\phi_T),$$

because the sign of  $\det(1 - T)$  is  $(-1)^k$  where  $k$  is the number (including multiplicities) of eigenvalues  $\lambda$  of  $T$  with  $|\lambda| > 1$ . Here  $\text{trace}_s(\phi_T)$  is the graded trace of the action of  $\phi_T$  on  $K^*(\mathbb{T}^n)$ , and  $P_n(\phi_T)$  is the number of periodic points of order  $n$ .

Putting things together, we see that

$$\frac{\det(1 - tX_+)}{\det(1 - tX_-)} = \exp(-1)^k \left( \sum_{n=1}^{\infty} P_n(\sigma) \frac{t^n}{n} \right).$$

The right-hand-side is called the *Artin-Mazur zeta function* of the map  $\phi_T$ .

To be explicit, if  $n = 2$  and  $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ , so  $k = 1$ ,  $X_+ = \text{Id}$  and  $X_-$  acts as  $T$  on  $K^1(\mathbb{T}^2) \cong \mathbb{C}^2$  and so  $\text{char}_{C(\mathbb{T}) \rtimes_{\phi_T} \mathbb{Z}}(t) = t^2 - t - 1$  and

$$\mathbb{T}\text{-spec}(C(\mathbb{T}^2) \rtimes_{\phi_T} \mathbb{Z}) = \left\{1, \frac{1 \pm \sqrt{5}}{2}\right\}.$$

Note that this yields another example of a  $\mathbb{T}$ - $C^*$ -algebra not  $\text{KK}^{\mathbb{T}}$ -equivalent to a commutative one.

Ian Putnam and his collaborators (see [24]) have studied a class of dynamical systems containing both this example and subshifts of finite type. These are called *Smale spaces*. Let  $X$  be such, so that  $X$  is equipped with a homeomorphism  $\sigma$  with 'hyperbolic' structure in the sense that, roughly, at each point of  $x$  there is a subspace  $V_x^u$  and a subspace  $V_x^s$  and a product structure  $X \cong V_x^s \times V_x^u$  in a neighbourhood of  $x$ , such that  $\phi$  is expanding along  $V_x^u$  and contracting along  $V_x^s$ .

Associated to this data are two groupoid  $C^*$ -algebras, the  $C^*$ -algebra of *stable equivalence*,  $S$ , and the  $C^*$ -algebra of *unstable equivalence*,  $U$ . The homeomorphism  $\sigma: X \rightarrow X$  induces automorphisms of  $S$  and  $U$ . The corresponding cross-products  $R^s := S \rtimes \mathbb{Z}$  and  $R^u := U \rtimes \mathbb{Z}$  are called the *Ruelle algebras* associated to the hyperbolic map. When  $(X, \sigma)$  is a topological Markov chain with transition matrix  $A$ , then  $S$  is an AF-algebra and  $R^s$  is strongly Morita equivalent to the corresponding Cuntz-Krieger algebra  $O_A$  and  $R^u$  is Morita equivalent to  $O_{A^T}$ .

Putnam has proved (see [25]) that  $K_*(S)$  has finite dimension and that the graded trace of  $\sigma_*$  acting on  $K_*(S)$  is equal to the number of fixed-points of  $\sigma: X \rightarrow X$ ; the same also is true for all positive powers of  $\sigma$ .

Now  $R^s$  is a  $\mathbb{T}$ -algebra, and by the same arguments as above we deduce that

$$\text{char}_{R^s}(t) = \zeta_{\text{AM}}(t).$$

To summarize, in certain examples of  $C^*$ -algebras  $A$  associated to hyperbolic dynamics, the spectral data involved in the  $\mathbb{C}[X, X^{-1}]$ -module  $K_*^{\mathbb{T}}(A)$  is sufficient to recover the Artin-Mazur zeta function of the original dynamical system.

#### 4. LOCALIZATION OF $\mathbb{T}$ -EQUIVARIANT KK-THEORY

If  $R$  is any commutative (unital) ring then any free, finitely generated  $R$ -module  $M$  has a well-defined *rank*, and any  $R$ -module self map of  $M$  has a well-defined *trace*. We denote these invariants by  $\text{rank}_R(M)$  and  $\text{trace}_R(L)$  respectively, so that in particular  $\text{trace}_R(\text{Id}) = \text{rank}_R(M)$ .

We will be mainly interested in the case where  $R = \mathbb{C}[X, X^{-1}]$  or a localization of  $R$ .

Consider  $\mathbb{C}[X, X^{-1}]$  as regular (rational) functions on  $\mathbb{C}^*$ . In algebraic geometry, if one wants to study the behavior of a variety near a point  $z \in \mathbb{C}^*$ , then one considers the set  $S$  of functions which are nonzero at  $z$ , and *localizes*  $\mathbb{C}[X, X^{-1}]$  with respect to this multiplicative set (a subset of a ring is a multiplicative set if it includes the unit 1 and is closed under multiplication.)

This means that we invert all functions which are in  $S$ , *i.e.* invert functions which do not vanish at  $z$ . We therefore get all rational functions which are regular at  $z$ .

$$\mathbb{C}[X, X^{-1}]_z \cong \{f \in \mathbb{C}(X) \mid f = \frac{h}{g}, g(z) \neq 0\}.$$

This is a *local ring*: it has a unique maximal ideal, the ideal of  $f \in \mathbb{C}[X, X^{-1}]_z$  such that  $f(z) = 0$ , and any  $f \in \mathbb{C}[X, X^{-1}]$  such that  $f(z) \neq 0$  is invertible in  $\mathbb{C}[X, X^{-1}]_z$ .

Note also that  $\mathbb{C}[X, X^{-1}]$  embeds in its localization.

Localization can be defined for any commutative ring  $R$  with no zero divisors, at a multiplicative subset  $S$  (like the complement of a prime ideal) by considering the elements  $\frac{r}{s}$  in the ring of fractions of  $R$ , such that  $s \in S$ . In this situation,  $R$  embeds in its localization.

For rings with zero divisors, localizations can still be defined, but the map from the original ring to its localization need not any longer be injective. Any element  $r \in R$  such that there exists  $s \in S$  so that  $rs = 0$ , is killed by localization at  $S$ .

The prime ideals of the localization of a ring  $R$  at  $S$  correspond to the prime ideals of  $R$  which do not intersect  $S$ .

The ‘localizations’  $\mathbb{C}[X, X^{-1}]_z$  just discussed, are the *stalks* of a sheaf of rings over  $\mathbb{C}^*$  (equipped with the Zariski topology. For most of this paper, we will not use the stalks, but the values of the sheaf on Zariski open sets. To fix notation and terminology, we state the definition formally.

**Definition 4.1.** Let  $f \in \mathbb{C}[X, X^{-1}]$ . The *localization* of  $\mathbb{C}[X, X^{-1}]$  at the Zariski open  $U_f := \mathbb{C}^* - Z_f$  is the ring obtained from  $\mathbb{C}[X, X^{-1}]$  by inverting all powers of  $f$ . We denote by  $\mathbb{C}[X, X^{-1}]_f$  the localization of  $\mathbb{C}[X, X^{-1}]$  at  $U_f$ . The assignment  $U_f \mapsto \mathbb{C}[X, X^{-1}]_f$  defines a sheaf on  $\mathbb{C}^*$  with the Zariski topology. The stalks of this sheaf are denoted  $\mathbb{C}[X, X^{-1}]_z$  and are as discussed above.

Note that inverting  $f$  automatically inverts all divisors of  $f$  and hence inverts all polynomials which do not vanish on  $U_f$ .

Hence  $\mathbb{C}[X, X^{-1}]_f$  is simply the ring of *regular rational functions on  $U_f$* .

Modules over a ring  $R$  can also be localized at multiplicative subsets of  $R$ , by setting

$$M_S := M \otimes_R R_S$$

where  $R_S$  is the localization of  $R$  at  $S$ . In the case of interest, where  $R = \mathbb{C}[X, X^{-1}]$ , we denote by  $M_f$  the localization of a  $\mathbb{C}[X, X^{-1}]$ -module at  $\{1, f, f^2, \dots\}$  (that is, at  $U_f$ .) The important point is that localization of a module at  $U_f$  kills torsion supported in  $Z_f$ . If  $M$  is a torsion module, thus with some finite support, then  $M_f = 0$  if  $f$  vanishes on the support. More generally, of course, if the support of the torsion module of a finitely generated module  $M$  is  $Z_f$  then localizing at  $U_f$  kills the torsion part, and the remainder is free (over  $\mathbb{C}[X, X^{-1}]_f$ .)

We now consider the case where the module  $M$  has the form  $M = K_{\mathbb{T}}^*(X)$  where  $X$  is a  $\mathbb{T}$ -space. More generally, we may consider any  $KK^{\mathbb{T}}$ -group, *i.e.* any  $KK_*^{\mathbb{T}}(A, B)$ , for  $A$  and  $B$   $\mathbb{T}$ - $C^*$ -algebras, with its  $\mathbb{C}[X, X^{-1}]$ -module structure. If  $f \in \mathbb{C}[X, X^{-1}]$  we may localize any such module at  $f$ , yielding  $KK_*^{\mathbb{T}}(A, B)_f$ . Localization is obviously compatible with the  $\mathbb{Z}/2$ -gradings, the intersection product (composition in  $KK^{\mathbb{T}}$ ) and the external product. In particular we may speak of  $KK_f^{\mathbb{T}}$ -equivalence and so on.

*Remark 4.2.* Localization in K-theory is slightly different from localization in equivariant cohomology as in [1].

- The coefficient ring  $\mathbb{C}[X, X^{-1}] = KK^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) = KK_*^{\mathbb{T}}(\mathbb{C}, \mathbb{C})$  we use is trivially graded, while the cohomological analogue  $H_{\mathbb{T}}^*(\text{pnt}) := H^*(B\mathbb{T}) \cong \mathbb{C}[u]$   $\mathbb{C}[u]$  is  $\mathbb{Z}$ -graded with  $\deg(u) = 2$ . Atiyah's Completion Theorem relates the two rings: equivariant cohomology is the  $I$ -adic completion of  $\mathbb{C}[X, X^{-1}]$  with respect to the ideal  $I$  corresponding to  $1 \in \mathbb{C}^*$ . Supports of  $\mathbb{C}[u]$ -modules, like for example  $H_{\mathbb{T}}^*(X) := H^*(E\mathbb{T} \times_{\mathbb{T}} X)$  for a  $\mathbb{T}$ -space  $X$ , are contained in  $\mathbb{C}$  instead of  $\mathbb{C}^*$ . If the modules are graded, then they are always either all of  $\mathbb{C}$  or are  $\{0\}$ , because they must be a cone (see [1]). Therefore the cohomological analogue of  $\mathbb{T}$ -spec is rather trivial: the support of the torsion submodule of  $H_{\mathbb{T}}^*(X)$  must be  $\{0\}$  and after localizing at  $\mathbb{C}^* := \mathbb{C} - \{0\}$  we get a free module.
- After localizing  $H_{\mathbb{T}}^*(X)$  by localizing, separately, its even and odd parts, the integer gradation on the module becomes lost; the  $\mathbb{Z}/2$ -grading is not lost, however.

Both of these facts would seem to support the idea that K-theory responds somewhat better to localization.

We are now going to refine some of our results from the first section about equivariant K-theory of spaces, using localization.

We begin by discussing the issue of finite generation, which, importantly, implies that the torsion sub-module of  $K_{\mathbb{T}}^*(X)$  has finite spectrum (see Example 4.5).

**Lemma 4.3.** ([29], Proposition 5.4). *If  $X$  is a smooth compact  $\mathbb{T}$ -manifold, then  $K_{\mathbb{T}}^*(X)$  is a finitely generated  $\mathbb{C}[X, X^{-1}]$ -module.*

The following is a useful geometric counterpart of Segal's lemma.

**Lemma 4.4.** *For a compact manifold  $X$  with smooth  $\mathbb{T}$ -action, there are only finitely many points  $t \in \mathbb{T}$  which fix some point of  $X - F$ .*

Hence if  $f \in \mathbb{C}[X, X^{-1}]$  is a polynomial which vanishes on these points, then  $K_{\mathbb{T}}^*(W \times X - F)_f = 0$  for any locally compact  $\mathbb{T}$ -space  $W$ .

*Proof.* For the first statement, since  $F$  is a smooth submanifold of  $X$  it has a normal bundle  $\nu$ , which is a  $\mathbb{T}$ -equivariant real vector bundle. This may be identified with the orthogonal complement of  $TF$  in  $TX|_F$  with respect to any  $\mathbb{T}$ -invariant Riemannian metric. Since the fixed-point set of  $t \in \mathbb{T}$  in  $T_x X$  (for  $x \in F$ ) is exactly  $TF$ ,  $t$  fixes no nonzero vector in  $\nu$ .

Let  $U$  be the corresponding  $\mathbb{T}$ -invariant open neighbourhood of  $F$ . Since  $\mathbb{T}$  acts freely on  $\nu - 0$  it acts freely on  $U - F$ . We can cover the compact  $X - U$  by finitely many open slices  $W_i \subset X - U$ , centred, say at points  $x_i$ , and if  $x \in X - U$  is any point, then  $\mathbb{T}_x \subset \mathbb{T}_{x_i}$  follows for  $x \in W_i$ . Since  $\mathbb{T}_x = \{1\}$  for  $x \in U$ ,  $\bigcup_{x \in X - F} \mathbb{T}_x$  is a finite set as claimed.

If  $(w, x) \in W \times X - F$  then of course  $\mathbb{T}_{(w, x)} \subset \mathbb{T}_x$ . It follows that if  $f \in \mathbb{C}[X, X^{-1}]$  vanishes on this set, then it annihilates the image of  $K_{\mathbb{T}}^*(Y) \rightarrow K_{\mathbb{T}}^*(W \times X - F)$  for any pre-compact  $Y \subset W \times X - F$ , *c.f.* the arguments in the first paragraph of the proof of Lemma 2.12. Hence it annihilates  $K_{\mathbb{T}}^*(W \times X - F)$ .  $\square$

*Example 4.5.* Consider  $\mathbb{T} \times \mathbb{N}$  with the  $\mathbb{T}$ -action of Example 2.8. Let  $X$  be the one-point compactification of  $X \times \mathbb{N}$ , with  $\mathbb{T}$ -action the canonical extension of the action on  $X \times \mathbb{N}$  (fixing the point at infinity). Then  $X$  is a compact space but there are infinitely many distinct points  $t \in \mathbb{T}$  which fix some point of  $X - F$ . The equivariant  $K$ -theory is zero in dimension 1 and in dimension 0 is the  $\mathbb{C}[X, X^{-1}]$ -module

$$K_{\mathbb{T}}^0(X) \cong \mathbb{C}[X, X^{-1}] \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}[X, X^{-1}]/(f_n)$$

where  $f_n(X) = \prod_{\omega \in \Omega_n} X - \omega$ . The torsion submodule of  $K_{\mathbb{T}}^0(X)$  is not finitely generated and has support  $\mathbb{C}^*$ . Thus both Lemma 4.3 and Lemma 4.4 fail for  $X$ .

**Lemma 4.6.** *Let  $A, B$  and  $C$  be  $\mathbb{C}[X, X^{-1}]$ -modules,  $\alpha: A \rightarrow B$  and  $\beta: B \rightarrow C$  module maps, such that the sequence*

$$0 \longrightarrow \text{im}(\alpha) \longrightarrow B \longrightarrow \ker(\beta) \longrightarrow 0$$

*is exact and  $A$  and  $C$  are finitely generated. Then  $B$  is finitely generated.*

*Proof.* This reduces immediately to whether  $\ker(\beta)$  and  $\text{im}(\alpha)$  are finitely generated; the latter is obvious and the former follows from the fact that any submodule of a finitely generated  $\mathbb{C}[X, X^{-1}]$ -module is finitely generated. (This property is equivalent, in general, to the Noetherian condition on  $\mathbb{C}[X, X^{-1}]$ .)  $\square$

**Corollary 4.7.**  *$K_{\mathbb{T}}^*(X - F)$  is finitely generated for any smooth and compact  $\mathbb{T}$ -manifold  $X$ .*

*Proof.* By Lemma 4.3  $K_{\mathbb{T}}^*(X)$  and  $K_{\mathbb{T}}^*(F)$  are finitely generated  $\mathbb{C}[X, X^{-1}]$ -modules. Now  $K_{\mathbb{T}}^*(X - F)$  fits into a 6-term exact sequence with the other terms  $K_{\mathbb{T}}^*(F)$  or  $K_{\mathbb{T}}^*(X)$  finitely generated so it follows that  $K_{\mathbb{T}}^*(X - F)$  is also finitely generated. It is a torsion module, and hence has a finite annihilator ideal and finite  $\mathbb{T}$ -spectrum (the finiteness of the spectrum also follows from induction and the geometric Lemma 4.4.)  $\square$

We will discuss  $\mathbb{T}$ -equivariant Poincaré duality for smooth manifolds in greater depth later; for now, the following statement is useful for proving certain things quickly.

**Theorem 4.8.** *Let  $X$  be a smooth and compact  $\mathbb{T}$ -manifold and  $D \in \mathrm{KK}^{\mathbb{T}}(C(\mathrm{TX}), \mathbb{C})$  the class of the Dirac operator on the almost-complex  $\mathbb{T}$ -manifold  $\mathrm{TX}$ . Then the cup-cap product with  $D$  determines a natural family of isomorphisms*

$$\mathrm{KK}_*^{\mathbb{T}}(C(X) \otimes A, B) \cong \mathrm{KK}_*^{\mathbb{T}}(A, C_0(\mathrm{TX}) \otimes B)$$

for all  $\mathbb{T}$ - $C^*$ -algebras  $A, B$ .

Theorem 4.8 is due to [9] in the non-equivariant setting. See also Kasparov [20] in the equivariant setting and his references. For a modern treatment of equivariant Poincaré duality see [15].

It follows from Poincaré duality that if  $W$  and  $Z$  are compact smooth  $\mathbb{T}$ -manifolds, then  $\mathrm{KK}_*^{\mathbb{T}}(C(W), C(Z))$  is a finitely generated  $\mathbb{C}[X, X^{-1}]$ -module. Indeed, duality reduces us to proving that  $K_{\mathbb{T}}^*(\mathrm{TW} \times Z)$  is finitely generated, which is easy to check (see Lemma 4.9). From this, and consideration of the 6-term exact sequence associated to  $F \subset X$ , we deduce that the morphism modules *e.g.*  $\mathrm{KK}_*^{\mathbb{T}}(C_0(X - F), C(F))$  between any two of  $C(X), C(F)$  and  $C_0(X - F)$  in  $\mathrm{KK}^{\mathbb{T}}$ , are finitely generated.

This statement definitely requires an assumption like smoothness; for example  $\mathrm{KK}_*^{\mathbb{T}}(C(W), C(Z))$  is not finitely generated if  $W$  is a Cantor set with trivial  $\mathbb{T}$ -action and  $Z$  is the one-point  $\mathbb{T}$ -space.

Smoothness also makes it much easier to prove the following (although this assumption could presumably be replaced by the hypothesis that  $K_{\mathbb{T}}^*(X)$  is finitely generated; we have not checked this.)

**Lemma 4.9.** *If  $X$  is a compact smooth  $\mathbb{T}$ -manifold and  $V \rightarrow X$  is a real  $\mathbb{T}$ -equivariant vector bundle on  $X$ , then  $K_{\mathbb{T}}^*(V)$  is finitely generated. Moreover, if  $\mathbb{T}$  acts fibrewise freely on  $V - 0$  then the restriction map*

$$K_{\mathbb{T}}^*(V) \rightarrow K_{\mathbb{T}}^*(X)$$

induces an isomorphism after localizing at  $\mathbb{C}^* - \{1\}$ .

In particular,  $\mathbb{T}\text{-spec}(V) = \mathbb{T}\text{-spec}(X) \cup \{1\}$  if  $\mathbb{T}$  acts freely on  $V - 0$ .

*Proof.* Fix a  $\mathbb{T}$ -invariant metric on  $V$  and consider the exact sequence

$$0 \longrightarrow C_0(D_V) \longrightarrow C_0(\overline{D}_V) \longrightarrow C(S_V) \longrightarrow 0$$

where  $D_V$  is the open disk bundle,  $\overline{D}_V$  the closed disk bundle, and  $S_V$  the sphere bundle. Since  $\overline{D}_V$  is  $\mathbb{T}$ -equivariantly proper homotopy equivalent to  $X$ , which is a compact smooth manifold, and since  $S_V$  is also a compact smooth manifold, it follows from considering the associated 6-term exact sequence and Lemma 4.6 that  $K_{\mathbb{T}}^*(V) \cong K_{\mathbb{T}}^*(D_V)$  is finitely generated.

If  $\mathbb{T}$  acts freely on  $V - 0$  then it acts freely on  $S_V$  and hence  $\mathbb{T}\text{-spec}(V) \subset \{1\}$ . Therefore, localizing at  $\mathbb{C}^* - \{1\}$  kills  $K_{\mathbb{T}}^*(S_V)$  and the claim follows.  $\square$

**Remark 4.10.** Suppose that  $V$  carries a  $\mathbb{T}$ -equivariant  $K$ -orientation. The *Euler class*  $e_V \in K^{-\dim(V)}(X)$  of  $V$  can be defined as the restriction to  $X$  (the zero section in  $V$ ) of the Thom class for  $V$ , in  $K_{\mathbb{T}}^{-\dim(V)}(V)$ . The Thom class generates  $K_{\mathbb{T}}^*(V)$  as a free rank-one  $K_{\mathbb{T}}^*(X)$ -module. It follows that the restriction map  $K_{\mathbb{T}}^*(V) \rightarrow K_{\mathbb{T}}^*(X)$  identifies, under  $K_{\mathbb{T}}^*(V) \cong K_{\mathbb{T}}^*(X)$ , with the map

$$K_{\mathbb{T}}^*(X) \rightarrow K_{\mathbb{T}}^*(X), \quad \xi \mapsto \xi \cdot e_V.$$

It follows then from Lemma 4.9 that  $e_V$  becomes an invertible after we localize at  $\mathbb{C}^* - \{1\}$ , that is,  $e_V$  is an invertible in the ring  $K_{\mathbb{T}}^*(X)_f$  where  $f(X) = X - 1$ .

This fact is used frequently in connection with characteristic class computations in the work of Atiyah and Segal and in Atiyah and Bott's paper [1].

**Theorem 4.11.** *Let  $X$  be a compact smooth  $\mathbb{T}$ -manifold and  $F \subset X$  the stationary set. Let*

$$\Omega := \{t \in \mathbb{T} \mid tx = x \text{ for some } x \in X - F\}.$$

*$\Omega$  is finite. Let  $f \in \mathbb{C}[X, X^{-1}]$  be a polynomial vanishing on  $\Omega$ . Then  $C_0(X - F)$  is  $\text{KK}_f^{\mathbb{T}}$ -equivalent to the zero  $\mathbb{T}$ - $C^*$ -algebra, and the localization  $\rho_f \in \text{KK}^{\mathbb{T}}(C(X), C(F))_f$  of the restriction morphism  $\rho \in \text{KK}^{\mathbb{T}}(C(X), C(F))$ , is invertible (in  $\text{KK}_f^{\mathbb{T}}$ ).*

*Proof.* To prove that  $C_0(X - F)$  is  $\text{KK}_f^{\mathbb{T}}$ -equivalent to zero it suffices to prove that  $\text{KK}_*^{\mathbb{T}}(C_0(X - F), C_0(X - F))_f$  is the zero module over  $\mathbb{C}[X, X^{-1}]_f$ . By the previous lemma, if  $f$  vanishes on  $\Omega$  then  $K_{\mathbb{T}}^*(\mathbb{T}F \times X - F)_f = 0 = K_{\mathbb{T}}^*(\mathbb{T}X \times X - F)_f$  and by Poincaré duality this implies that

$$(4.1) \quad \text{KK}_*^{\mathbb{T}}(C(F), C_0(X - F))_f = 0, \quad \text{KK}_*^{\mathbb{T}}(C(X), C_0(X - F))_f = 0.$$

Using the 6-term exact sequence applied to the first variable, we deduce that

$$\text{KK}_*^{\mathbb{T}}(C_0(X - F), C_0(X - F))_f = 0$$

too. Thus,  $C_0(X - F)$  is  $\text{KK}_f^{\mathbb{T}}$ -equivalent to the zero  $\mathbb{T}$ - $C^*$ -algebra as claimed.

(We could not use Poincaré duality directly for  $C_0(X - F)$  because it is non-compact, and duality works differently for non-compact spaces.)

Now from the 6-term exact sequence, and the fact just proved that  $C_0(X - F)$  is  $\text{KK}_f^{\mathbb{T}}$ -equivalent to zero, the map

$$(4.2) \quad \text{KK}_*^{\mathbb{T}}(A, C(X))_f \xrightarrow{\cdot \otimes_{C(X)} \rho} \text{KK}_*^{\mathbb{T}}(A, C(F))_f$$

induced by restriction to  $F$  is an isomorphism for any  $\mathbb{T}$ - $C^*$ -algebra  $A$ . Now use the Yoneda lemma: set  $A := C(F)$  and find a pre-image  $\alpha \in \text{KK}^{\mathbb{T}}(C(F), C(X))_f$  of the identity morphism in  $\text{KK}_*^{\mathbb{T}}(C(F), C(F))_f$ . Then the composition in  $\text{KK}_f^{\mathbb{T}}$

$$C(F) \xrightarrow{\alpha} C(X) \xrightarrow{\rho} C(F)$$

is the identity by the definitions, and the composition

$$C(X) \xrightarrow{\rho} C(F) \xrightarrow{\alpha} C(X)$$

is therefore multiplication by an idempotent  $\gamma := \rho \otimes_{C(F)} \alpha \in \text{KK}^{\mathbb{T}}(C(X), C(X))_f$ . To show that  $1 - \gamma = 0$  set  $A := C(X)$ , and observe that this is mapped to zero under composition with  $\rho$ , *i.e.* under the map (4.2). Since the latter is an isomorphism after localization,  $1 - \gamma = 0$ . □

*Remark 4.12.* While a properly formulated version of Theorem 4.11 *should* be true without smoothness assumptions (*c.f.* Theorem 2.5, which does not use such an assumption), we have not pursued it since we are mainly interested in smooth manifolds anyway, and because Example 4.5 shows that away from smooth manifolds,  $\mathbb{T}\text{-spec}(X - F)$  may not be finite, which makes it more difficult to formulate a theorem.



We note also that an alternative, and more powerful proof technique for Theorem 4.11, is to use adjointness of induction and restriction functors in KK and an inductive argument to prove directly that  $\mathrm{KK}^{\mathbb{T}}(C_0(X - F), C_0(X - F)) = 0$ . This eliminates the need to use duality. Since however we need duality anyway, we have chosen to take this route.

Our work gives us the following Lemma, which is essential for proving the Lefschetz theorem in the next section.

**Lemma 4.13.** *Let  $D$  be a  $\mathbb{T}$ - $C^*$ -algebra in the bootstrap category, such that  $D \rtimes \mathbb{T}$  is also in the bootstrap category, let  $X$  be a smooth, compact  $\mathbb{T}$ -manifold, and  $f, \Omega$  be as in Theorem 4.11. Then*

- $\mathrm{KK}_*^{\mathbb{T}}(C(X), D)_f \cong \mathrm{Hom}_{\mathbb{C}[X, X^{-1}]_f}(\mathrm{K}_{\mathbb{T}}^*(X)_f, \mathrm{K}_{\mathbb{T}}^*(D)_f)$
- $\mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, C(X) \otimes D)_f \cong \mathrm{K}_{\mathbb{T}}^*(X) \otimes_{\mathbb{C}[X, X^{-1}]_f} \mathrm{K}_{\mathbb{T}}^*(D)$ .

*Proof.* The class of  $\mathbb{T}$ -spaces  $X$  for which both theorems hold (in  $\mathrm{KK}_f^{\mathbb{T}}$ ) is closed under  $\mathrm{KK}_f^{\mathbb{T}}$ -equivalence so we may replace  $X$  by  $F$ ; both results then follow trivially.  $\square$

We end this section with a fairly precise description of  $\mathrm{K}_{\mathbb{T}}^*(X)$  for smooth  $\mathbb{T}$ -manifolds, starting with the following result, which uses a theorem of Baum and Connes (see [5]).

**Theorem 4.14.** *Let  $X$  be a smooth, compact  $\mathbb{T}$ -manifold,  $F \subset X$  the stationary set.*

*For  $\gamma \in \mathbb{T}$  we endow the  $\mathbb{C}$ -vector space  $\mathrm{K}^*(\mathbb{T} \setminus X^\gamma - F)$  with the  $\mathbb{C}[X, X^{-1}]$ -module structure by evaluation  $\mathbb{C}[X, X^{-1}] \rightarrow \mathbb{C}$  at  $\gamma$ . Then*

$$(4.3) \quad \mathrm{K}_{\mathbb{T}}^*(X - F) \cong \bigoplus_{\gamma \in \mathbb{T}\text{-spec}(X - F)} \mathrm{K}^*(\mathbb{T} \setminus X^\gamma - F)$$

*as  $\mathbb{C}[X, X^{-1}]$ -modules.*

*Remark 4.15.* The usual geometric effect of localization of  $\mathrm{K}_{\mathbb{T}}^*(X)$  at  $\gamma \in \mathbb{T}$  – it annihilates the contribution of  $X - X^\gamma$ , as we have seen – is obviously nil in the case where  $\gamma = 1$ . Thus Theorem 4.14 goes further in this case, informing us that the stalk at 1 of the sheaf determined by  $\mathrm{K}_{\mathbb{T}}^*(X - F)$  is  $\mathrm{K}^*(\mathbb{T} \setminus X - F)$  (with  $\mathbb{C}[X, X^{-1}]$ -module structure by evaluation at  $1 \in \mathbb{C}^*$ .)

In particular, we now have an exact description of  $\mathbb{T}\text{-spec}(X)$  when  $X$  is a compact smooth manifold.

**Corollary 4.16.** *Let  $X$  be a compact smooth  $\mathbb{T}$ -manifold with no stationary points. Then*

$$\mathbb{T}\text{-spec}(X) = \{\gamma \in \mathbb{T} \mid \mathrm{K}^*(\mathbb{T} \setminus X^\gamma) \neq 0\}.$$

Before the proof, we use Theorem 4.14 to determine the exact relation between the torsion submodule of  $\mathrm{K}_{\mathbb{T}}^*(X)$  and the torsion module  $\mathrm{K}_{\mathbb{T}}^*(X - F)$ .

Let  $\mathrm{Tors}(\mathrm{K}_{\mathbb{T}}^i(X))$  be the torsion part of  $\mathrm{K}_{\mathbb{T}}^i(X)$  and  $\mathrm{Free}(\mathrm{K}_{\mathbb{T}}^i(X))$  the free part. The 6-term exact sequence associated to the stationary set  $F \subset X$  yields *surjections*

$$\mathrm{K}_{\mathbb{T}}^i(X \setminus F) \rightarrow \mathrm{Tors}(\mathrm{K}_{\mathbb{T}}^i(X))$$

since the map  $\mathrm{K}_{\mathbb{T}}^i(X) \rightarrow \mathrm{K}_{\mathbb{T}}^i(F)$  vanishes on the torsion part, since  $\mathrm{K}_{\mathbb{T}}^*(F)$  is free, and *injections*

$$\mathrm{Free}(\mathrm{K}_{\mathbb{T}}^i(X)) \rightarrow \mathrm{K}_{\mathbb{T}}^i(F) \cong \mathrm{K}^i(F) \otimes \mathbb{C}[X, X^{-1}],$$

since the map  $K_{\mathbb{T}}^{i+1}(X - F) \rightarrow K_{\mathbb{T}}^{i+1}(X)$  has range in the torsion subgroup.

We have the boundary maps

$$(4.4) \quad \partial_i: K^{i-1}(F) \otimes \mathbb{C}[X, X^{-1}] \longrightarrow K_{\mathbb{T}}^i(X - F)$$

and thus

$$\text{coker}(\partial_i) \cong \text{Tors}(K_{\mathbb{T}}^i(X)), \quad \ker(\partial_{i+1}) \cong \text{Free}(K_{\mathbb{T}}^i(X)).$$

Theorem 4.14 and some geometric arguments (using smoothness) tells us more.

**Corollary 4.17.** *If  $X$  is a smooth compact  $\mathbb{T}$ -manifold, then the range of  $\partial_i: K_{\mathbb{T}}^i(F) \rightarrow K_{\mathbb{T}}^{i+1}(X - F)$  is supported at  $1 \in \mathbb{C}^*$ . Hence  $\partial_i$  factors through a map*

$$\partial'_i: K_{\mathbb{T}}^i(F) \rightarrow K^{i+1}(\mathbb{T} \setminus X - F).$$

*Then  $\text{Tors}(K_{\mathbb{T}}^i(X))_z \cong K_{\mathbb{T}}^i(X - F)_z$  for all  $z \in \mathbb{T} - \{1\}$ , and for the component at  $1 \in \mathbb{C}^*$  we have*

$$\text{Tors}(K_{\mathbb{T}}^i(X)_1) \cong K^i(\mathbb{T} \setminus X - F) / \text{im}(\partial'_{i+1}).$$

*In other words, the torsion modules  $\text{Tors}(K_{\mathbb{T}}^*(X))$  and  $K_{\mathbb{T}}^*(X - F)$  agree away from  $1 \in \mathbb{C}^*$ .*

*Remark 4.18.* This result is somewhat comparable to the computation of *integral* equivariant  $K$ -theory for proper co-compact discrete group actions, in [10].

The boundary maps in the 6-term exact sequence of Theorem 4.14 can be computed fairly precisely if  $X$  is a smooth manifold with smooth  $\mathbb{T}$ -action, and this also proves the Corollary 4.17.

For the definition of correspondence, used below, see the discussion in §5.

*Proof.* (Of Corollary 4.17).  $F$  is a closed, smooth submanifold of  $X$ . Let  $\nu$  be the normal bundle of the stationary set  $F \subset X$ ; it can be endowed with a  $\mathbb{T}$ -action and invariant Riemannian metric. Let  $\hat{\varphi}: \nu \rightarrow X$  the tubular neighbourhood embedding.

Let  $S\nu$  be the sphere bundle of  $\nu$  and  $\pi: \nu \rightarrow F$  the bundle projection. Let  $j: S\nu \rightarrow X$  be its restriction to  $S\nu$ . Note that  $j(S\nu)$  is disjoint from  $F$  and that  $j$  is a canonically  $\mathbb{T}$ -equivariantly  $K$ -oriented embedding with trivial normal bundle. To see this, define

$$\hat{f}: S\nu \times \mathbb{R} \cong U_F \subset X - F, \quad \hat{f}(x, \xi, s) := \hat{\varphi}(s\xi).$$

The restriction of  $\hat{f}$  to the zero section  $S\nu \times \{0\}$  is the embedding  $j$ .

The class in  $\text{KK}_{\mathbb{T}}^1(C(F), C_0(X - F))$  of the  $\mathbb{T}$ -equivariant extension

$$0 \longrightarrow C_0(X - F) \longrightarrow C(X) \longrightarrow C(F) \longrightarrow 0$$

is equal (see [9] Proposition 3.6.; the equivariant version goes through in the same way since we have a  $\mathbb{T}$ -equivariant normal bundle) to the class of the  $\mathbb{T}$ -equivariant correspondence

$$S\nu \xleftarrow{\pi_{S\nu}} (S\nu \times \mathbb{R}, \beta_{\mathbb{R}}) \xrightarrow{\hat{f}} X - F$$

where  $\beta_{\mathbb{R}} \in K_{\mathbb{T}}^1(\mathbb{R})$  is the Bott class (for the trivial  $\mathbb{T}$ -action on  $\mathbb{R}$ .)

Hence the class  $\partial[V] \in K_{\mathbb{T}}^1(X - F)$  is then represented by the smooth  $\mathbb{T}$ -equivariant correspondence  $\text{pnt} \leftarrow (S\nu, \pi_{S\nu}^*(V)) \xrightarrow{j} X \setminus F$ , alternatively, as the class

$$\hat{f}_!(\pi_{S\nu}^*(V) \cdot \beta_{\mathbb{R}}) \in K_{\mathbb{T}}^1(X - F)$$

of the Thom class of the (trivial) normal bundle, pushed forward to  $X - F$  via  $\hat{f}$ .

Note that since  $\mathbb{T}$  acts freely on  $\nu - 0$ , the open neighbourhood  $U_F$  of  $\hat{\varphi}(S\nu)$  may be assumed to meet none of the  $X^\gamma$  with  $\gamma \in \mathbb{T} - \{1\}$ . Hence localizing at  $\gamma \neq 1$  kills the range of  $\partial_0$ , so its range is contained in the component of  $K_{\mathbb{T}}^1(X - F)$  over  $1 \in \mathbb{T}$ . Similarly for  $i = 1$ .  $\square$

*Remark 4.19.* We make several remarks about the proof.

- We can describe the maps  $\partial_i'$  more precisely. In the proof of Corollary 4.17 we observed that there is a  $\mathbb{T}$ -equivariant correspondence

$$S\nu \xleftarrow{\pi_{S\nu}} (S\nu \times \mathbb{R}, \beta_{\mathbb{R}}) \xrightarrow{\hat{f}} X - F.$$

In fact by shrinking the neighbourhood  $U_F$  of  $S\nu$  if needed so that it is disjoint from  $F$ , we can factor  $\hat{f}$  through an open embedding  $\hat{f}': S\nu \times \mathbb{R} \rightarrow U_F - F$  and the open embedding  $U_F - F \rightarrow X - F$ . The first yields a class in  $KK_{\mathbb{T}}^1(C(S\nu), C_0(U_F - F))$  but this group maps, using descent, to  $KK_{\mathbb{T}}^1(C(\mathbb{T} \backslash S\nu), C_0(U_F - F))$  since  $\mathbb{T}$  acts freely on  $S\nu$  and  $U_F - F$ . Now the open embedding  $U_F - F \rightarrow X - F$  induces an open embedding of quotient spaces  $\mathbb{T} \backslash U_F - F \rightarrow \mathbb{T} \backslash X - F$  and an element  $j! \in KK(C_0(\mathbb{T} \backslash U_F - F), C_0(\mathbb{T} \backslash X - F))$ . The map  $\partial_i'$  is the composition

$$(4.5) \quad K_{\mathbb{T}}^i(F) \xrightarrow{\pi_{S\nu}^*} K_{\mathbb{T}}^i(S\nu) \cong K^i(\mathbb{T} \backslash S\nu) \xrightarrow{\hat{f}'} K^{i+1}(U_F - F) \xrightarrow{j!} K^{i+1}(\mathbb{T} \backslash X - F).$$

- The boundary map  $\partial_0: K^0(F) \otimes \mathbb{C}[X, X^{-1}] \rightarrow K_{\mathbb{T}}^1(X - F)$  may be understood as giving an obstruction to extending a  $\mathbb{T}$ -equivariant vector bundle on  $F$  to a  $\mathbb{T}$ -equivariant vector bundle on  $X$ : this is possible for a given  $[V]$  if and only if  $\partial_0[V] = 0$ , which is if and only if the class

$$\hat{f}_! (\pi_{S\nu}^*(V) \cdot \beta_{\mathbb{R}}) \in K^1(\mathbb{T} \backslash X - F)$$

vanishes. The obstruction group for the extension problem, in other words, is  $K^0(F) \otimes \mathbb{C}[X, X^{-1}] / \ker(\partial_0)$ .

*Proof.* (Of Theorem 4.14). For the following discussion it is better to consider the stalks  $K_{\mathbb{T}}^*(X)_{\gamma}$  of the sheaf  $U_f \mapsto K_{\mathbb{T}}^*(X)_f$ . If – as in the case in the following Lemma – a  $\mathbb{T}$ -space  $X$  has finite spectrum, then these stalks can be described simply as follows. For  $\gamma \in \mathbb{T}\text{-spec}(X) \subset \mathbb{T}$ , set

$$f_{\gamma} = \prod_{\gamma' \in \mathbb{T}\text{-spec}(X) - \{\gamma\}} (X - \gamma').$$

Then  $K_{\mathbb{T}}^*(X)_{\gamma} = K_{\mathbb{T}}^*(X)_{f_{\gamma}}$ .

It should be clear at this stage that  $K_{\mathbb{T}}^*(X - X^{\gamma})_{\gamma} = 0$  for any  $\gamma \in \mathbb{T}$ , since all contributions to the K-theory of  $X - X^{\gamma}$  from open slices induced from isotropy groups at points  $x \in X - X^{\gamma}$  are supported away from  $\gamma$ , since  $\gamma \notin \mathbb{T}_x$ . From the 6-term exact sequence, we get

$$K_{\mathbb{T}}^*(X)_{\gamma} \cong K_{\mathbb{T}}^*(X^{\gamma})_{\gamma}$$

and the module structure on the right-hand-side is evaluation of polynomials at  $\gamma \in \mathbb{T}$ .

Since

$$K_{\mathbb{T}}^*(X) \cong \bigoplus_{\gamma \in \mathbb{T}\text{-spec}(X)} K_{\mathbb{T}}^*(X)_{\gamma}$$

it remains to analyse the  $\mathbb{C}[X, X^{-1}]$ -modules

$$K_{\mathbb{T}}^*(X)_{\gamma} \cong K_{\mathbb{T}}^*(X^{\gamma})_{\gamma} := K_{\mathbb{T}}^*(X^{\gamma}) \otimes_{\mathbb{C}[X, X^{-1}]} \mathbb{C}[X, X^{-1}]_{\gamma}.$$

Consider two sheaves of  $\mathbb{C}[X, X^{-1}]_{\gamma}$ -modules on  $X^{\gamma}$ . The first assigns to an open  $\mathbb{T}$ -invariant subset  $U \subset X^{\gamma}$  the module  $K^*(\mathbb{T} \backslash U)$  with module structure evaluation of  $f \in \mathbb{C}[X, X^{-1}]_{\gamma}$  at  $\gamma$ . The second assigns to  $U \subset X^{\gamma}$  the module  $K_{\mathbb{T}}^*(U)_{\gamma}$ . We claim that they agree on open slices.

Let  $U \cong \mathbb{T} \times_H Y$  be such an open slice. Then  $K^*(\mathbb{T} \backslash U) \cong K^*(H \backslash Y)$  gives the value of the first sheaf. For the second, we appeal to a result (see [5]) of Paul Baum and Alain Connes for equivariant K-theory for *finite* group actions.

We have  $K_{\mathbb{T}}^*(U) \cong K_H^*(Y)$  as  $\mathbb{C}[X, X^{-1}]$ -modules, where the  $\mathbb{C}[X, X^{-1}]$ -module action on  $K_H^*(Y)$  factors through the restriction  $\mathbb{C}[X, X^{-1}] \rightarrow \text{Rep}(H)$  and the  $\text{Rep}(H)$ -module structure on  $K_H^*(Y)$ . By the result of Baum and Connes,

$$K_H^*(Y) \cong \bigoplus_{h \in H} K^*(H \backslash Y^h),$$

where the  $\text{Rep}(H) \cong \mathbb{C}[X, X^{-1}]/(f_H)$ -module structure on the right-hand-side is by evaluation of characters at the points of  $H$  (here  $f_H = \prod_{h \in H} X - h$  and  $(f_H)$  is the ideal of  $\mathbb{C}[X, X^{-1}]$  generated by  $f_H$ .) We are using the fact that  $H$  is abelian, so that the centralizer of  $\gamma$  in  $H$  is  $H$ . Hence

$$K_H^*(Y)_{\gamma} := K_H^*(Y) \otimes_{\mathbb{C}[X, X^{-1}]} \mathbb{C}[X, X^{-1}]_{\gamma} \cong \bigoplus_{h \in H} [K^*(H \backslash Y^h) \otimes_{\mathbb{C}[X, X^{-1}]} \mathbb{C}[X, X^{-1}]_{\gamma}].$$

Now for each term on the right-hand-side, the tensor product is over the evaluation map  $\mathbb{C}[X, X^{-1}] \rightarrow \mathbb{C}$  at  $h$ . It follows that all terms in the sum on the right-hand-side vanish except for  $h = \gamma$ . The  $\mathbb{C}[X, X^{-1}]_{\gamma}$ -module structure on this term is evaluation of polynomials at  $\gamma$ . Observe also that  $Y^{\gamma} = Y$ , since  $Y \subset X^{\gamma}$ . Therefore

$$K_H^*(Y)_{\gamma} \cong K^*(H \backslash Y).$$

and the two sheaves agree on open slices, as claimed. The result now follows from a standard spectral sequence argument.  $\square$

*Remark 4.20.* It would be interesting to study further the maps  $\partial'_i$  and the obstruction groups  $K^i(\mathbb{T} \backslash X - F) / \text{im}(\partial'_i)$  but we do not pursue this further now.

## 5. THE LEFSCHETZ THEOREM

**Definition 5.1.** Let  $X$  be a smooth, compact  $\mathbb{T}$ -manifold. Let

- $D \in \text{KK}_0^{\mathbb{T}}(C_0(\text{TX}), \mathbb{C})$  be the class of the  $\mathbb{T}$ -equivariant Dirac operator on the almost-complex manifold  $\text{TX}$ .
- $\Theta \in \text{KK}_0^{\mathbb{T}}(C_0(X), C_0(X \times \text{TX}))$  the class of the  $\mathbb{T}$ -equivariant K-oriented embedding  $\rho: X \rightarrow X \times \text{TX}$ ,  $\rho(x) := (x, (x, 0))$ .
- $s$  be the proper  $\mathbb{T}$ -map  $\text{TX} \rightarrow X \times \text{TX}$ ,  $s(x, \xi) := ((x, \xi), x)$ .

Then the *Lefschetz map* (see [15])

$$\text{Lef}: \text{KK}_*^{\mathbb{T}}(C(X), C(X)) \rightarrow \text{KK}_*^{\mathbb{T}}(C(X), \mathbb{C})$$

is the composition

$$(5.1) \quad \begin{aligned} \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X)) &\xrightarrow{\otimes c^{1_{TX}}} \mathrm{KK}_*^{\mathbb{T}}(C_0(X \times TX), C_0(X \times TX)) \\ &\xrightarrow{s^*} \mathrm{KK}_*^{\mathbb{T}}(C_0(X \times TX), C_0(TX)) \\ &\xrightarrow{\otimes_{C_0(TX)} D} \mathrm{KK}_*^{\mathbb{T}}(C_0(X \times TX), \mathbb{C}) \xrightarrow{\Theta \otimes_{C_0(X \times TX)}} \mathrm{KK}_*^{\mathbb{T}}(C(X), \mathbb{C}) \end{aligned}$$

Thus the Lefschetz map associates to an equivariant morphism  $X \rightarrow X$  in  $\mathrm{KK}^{\mathbb{T}}$ , an equivariant K-homology class for  $X$ . Such a class has an index in  $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$ .

**Definition 5.2.** The *Lefschetz index*  $\mathrm{Ind}^{\mathrm{Lef}}(\Lambda)$ , where  $f \in \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$  is the  $\mathbb{T}$ -equivariant index

$$\mathrm{Ind}^{\mathrm{Lef}}(\Lambda) := (\mathrm{pnt})_* \mathrm{Lef}(\Lambda) \in \mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}],$$

where  $\mathrm{pnt}: X \rightarrow \mathrm{pnt}$  is the map to a point.

In [17] and [18] we proved that  $\mathbb{T}$ -equivariant correspondences are cycles for a bivariant homology theory isomorphic to  $\mathrm{KK}^{\mathbb{T}}$ , with some restrictions on its arguments (*e.g.* to compact smooth  $\mathbb{T}$ -manifolds.)

Hence both the domain and co-domain of the Lefschetz map can be described in terms of equivalence classes of correspondences; since we have defined the Lefschetz map itself in terms of correspondences, the Lefschetz map can be described in purely geometric terms. We give a brief summary.

Suppose the following data is given (see the original reference [9]), or [17].)

- $M$  is a smooth  $\mathbb{T}$ -manifold (not necessarily compact).
- $b: M \rightarrow X$  is a smooth  $\mathbb{T}$ -map (not necessarily proper).
- $\xi \in \mathrm{RK}_{\mathbb{T}, X}^*(M)$  is an equivariant K-theory class with compact support along the fibres of  $b$ .
- $f: M \rightarrow X$  is a  $\mathbb{T}$ -equivariant smooth K-oriented map.

This data is sometimes summarized by a diagram  $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ . The quadruple  $(M, b, f, \xi)$  is a  $\mathbb{T}$ -equivariant correspondence from  $X$  to  $X$ .

It is convenient to assume that the correspondence – denote it  $\Lambda$  – also satisfies

- $f: M \rightarrow X$  is a submersion.
- The map  $X \rightarrow X \times X$ ,  $x \mapsto (f(x), b(x))$  is transverse to the diagonal  $X \rightarrow X \times X$ .

These conditions imply that the *coincidence space*  $F := \{x \in M \mid f(x) = g(x)\}$  has the structure of a smooth, equivariantly K-oriented  $\mathbb{T}$ -manifold (probably disconnected, but with only finitely many connected components, but each of the same dimension.)

Clearly it comes with a map  $b|_F: F \rightarrow X$ , so we obtain a Baum-Douglas cycle  $(F, b|_F, \xi|_F)$  for  $X$  by restricting  $\xi$  to  $F \subset M$ .

To a correspondence is associated a class, which by abuse of notation we also denote by  $\Lambda$ , in  $\mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$ . Here  $*$  =  $\dim(M) - \dim(X) + \dim(\xi)$ . See [17] for the details.

The following is a straightforward manipulation with correspondences.

**Proposition 5.3.** *If  $\Lambda \in \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$  is represented by the  $\mathbb{T}$ -equivariant correspondence in general position in the sense described above, then  $\mathrm{Lef}(\Lambda)$  is represented by the Baum-Douglas cycle  $(F, b|_F, \xi|_F)$  for  $X$ . In particular,*

$$\mathrm{Ind}^{\mathrm{Lef}}(\Lambda) = \mathrm{ind}_{\mathbb{T}}(D_F \cdot \xi) \in \mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$$

*holds; that is, the Lefschetz index of  $\Lambda$  equals the  $\mathbb{T}$ -index of the  $\mathbb{T}$ -equivariant Dirac operator on the coincidence manifold  $F$ , twisted by  $\xi|_F$ .*

We will not prove this proposition; the proof can be found in [16] or the reader reasonably familiar with correspondences can prove it himself.

We aim to prove that  $\mathrm{Ind}^{\mathrm{Lef}}(\Lambda) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_*)$  for where  $\Lambda_*: K_{\mathbb{T}}^*(X) \rightarrow K_{\mathbb{T}}^*(X)$  is the action of  $\Lambda$  on equivariant K-theory; note that  $\Lambda_*$  is a  $\mathbb{C}[X, X^{-1}]$ -module map. Proving this statement has nothing to do with correspondences; it depends only on formal properties of  $\mathrm{KK}^{\mathbb{T}}$ .

The result provides a homological interpretation of the Lefschetz index along the lines of the classical theorem.

By the *trace* we mean the following. Firstly, since  $X$  is a smooth compact manifold,  $K_{\mathbb{T}}^*(X)$  is a finitely generated  $\mathbb{C}[X, X^{-1}]$ -module. Therefore (in each dimension  $*$  = 0, 1 it decomposes into a free part and a torsion part. Any  $\mathbb{C}[X, X^{-1}]$ -module self-map of  $K_{\mathbb{T}}^*(X)$  of even degree will induce a grading-preserving map on K-theory. We will define the trace of such a map to be the differences of the  $\mathbb{C}[X, X^{-1}]$ -valued module traces on  $K_{\mathbb{T}}^0(X)$  and  $K_{\mathbb{T}}^1(X)$ . To define these individually, consider any  $\mathbb{C}[X, X^{-1}]$  module, which we write as  $M = T \oplus \mathbb{C}[X, X^{-1}]^k$  where  $T$  is torsion. Any self  $\mathbb{C}[X, X^{-1}]$ -module map of  $M$  sends  $T$  to itself and hence has an upper-triangular form  $L = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$  and we let  $\mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(L) := \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(C)$ . This is uniquely defined.

A  $\mathbb{C}[X, X^{-1}]$ -module self-map of  $K_{\mathbb{T}}^*(X)$  with odd degree will have trace zero, by definition.

**Theorem 5.4.** *(Lefschetz theorem in  $\mathrm{KK}^{\mathbb{T}}$ ). Let  $X$  be a compact smooth  $\mathbb{T}$ -manifold and  $\Lambda \in \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$ . Then  $\mathrm{Ind}^{\mathrm{Lef}}(\Lambda) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_*)$ .*

Before proceeding, note that since  $\mathrm{Lef}$  (and  $\mathrm{Ind}^{\mathrm{Lef}}$ ) are both defined by basic  $\mathrm{KK}^{\mathbb{T}}$ -operations, both maps are compatible in the obvious sense with localization. For any  $A$  and  $B$  and any  $\alpha \in \mathrm{KK}_*^{\mathbb{T}}(A, B)$ , and any  $f \in \mathbb{C}[X, X^{-1}]$ , denote by  $\alpha_f \in \mathrm{KK}_*^{\mathbb{T}}(A, B)_f$  the image of  $f$  under localization at  $U_f$ . Then compatibility means that the diagram

$$(5.2) \quad \begin{array}{ccccc} \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X)) & \xrightarrow{\mathrm{Lef}} & \mathrm{KK}_*^{\mathbb{T}}(C(X), \mathbb{C}) & \xrightarrow{\mathrm{ind}_{\mathbb{T}}} & \mathbb{C}[X, X^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))_f & \xrightarrow{\mathrm{Lef}} & \mathrm{KK}_*^{\mathbb{T}}(C(X), \mathbb{C})_f & \xrightarrow{\mathrm{ind}_{\mathbb{T}}} & \mathbb{C}[X, X^{-1}]_f \end{array}$$

commutes, where the lower row is the ‘localized’ Lefschetz index map, defined using Kasparov products as on the top row, except with the localized classes  $D_f, \Theta_f$  and so on.

Neither the first nor second vertical map need be injective, of course, but the third vertical map is injective because  $\mathbb{C}[X, X^{-1}]$  is an integral domain. The diagram says that  $\text{Ind}^{\text{Lef}}(\Lambda)_f = \text{Ind}_f^{\text{Lef}}(\Lambda_f)$  where  $\text{Ind}_f^{\text{Lef}}$  is the Lefschetz map in localized  $\text{KK}^{\mathbb{T}}$ .

We define the localized module trace

$$\text{trace}_{\mathbb{C}[X, X^{-1}]_f} : \text{End}_{\mathbb{C}[X, X^{-1}]_f}(K_{\mathbb{T}}^*(X)_f) \rightarrow \mathbb{C}[X, X^{-1}]_f$$

as with the non-localized version. Note that localization of a  $\mathbb{C}[X, X^{-1}]$ -module respects the decomposition into its torsion and free parts, so that

$$(5.3) \quad \text{trace}_{\mathbb{C}[X, X^{-1}]_f}(L_f) = [\text{trace}_{\mathbb{C}[X, X^{-1}]}(L)]_f$$

is clear, for any  $\mathbb{C}[X, X^{-1}]$ -module self-map of  $K_{\mathbb{T}}^*(X)$ .

It will be sufficient to prove the following apparently weaker version of Theorem 5.4.

**Lemma 5.5.** *Let  $\Omega$  be as in Theorem 4.11 and  $f \in \mathbb{C}[X, X^{-1}]$  vanish on  $\Omega$ . Then the Lefschetz theorem for  $X$  holds in  $\text{KK}_f^{\mathbb{T}}$ . That is,*

$$\text{Ind}_f^{\text{Lef}}(\Lambda_f) = \text{trace}_{\mathbb{C}[X, X^{-1}]_f}((\Lambda_f)_*)$$

for any  $\Lambda \in \text{KK}_*^{\mathbb{T}}(C(X), C(X))$ .

Lemma 5.5 implies Theorem 5.4 because combining the diagram (5.2) and its algebraic analogue (5.3) gives

$$(5.4) \quad \begin{aligned} \text{Ind}^{\text{Lef}}(\Lambda)_f &= \text{Ind}_f^{\text{Lef}}(\Lambda_f) = \text{trace}_{\mathbb{C}[X, X^{-1}]_f}((\Lambda_f)_*) \\ &= [\text{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda)]_f \in \mathbb{C}[X, X^{-1}]_f. \end{aligned}$$

By injectivity of  $\mathbb{C}[X, X^{-1}] \rightarrow \mathbb{C}[X, X^{-1}]_f$ , it follows that  $\text{Ind}^{\text{Lef}}(\Lambda) = \text{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_*)$ , yielding Theorem 5.4.

To prove Lemma 5.5 it is useful to use a slightly different formalism for the Lefschetz *indices*  $\text{Ind}^{\text{Lef}}(\cdot)$ . This formalism is more general in the sense that it applies to noncommutative  $\mathbb{T}$ - $C^*$ -algebras as well, provided they have duals. (The Lefschetz *map* of Definition 5.1 exists in more generality than we have suggested, but does not work for noncommutative algebras because of the implicit use of the ‘diagonal map’  $X \rightarrow X \times TX$ .)

As above,  $s: TX \rightarrow X \times TX$  is the obvious section. Let  $\Sigma: X \times TX \rightarrow TX \times X$  be the flip. Set

- $\Delta := \Sigma^* s_*(D) \in \text{KK}^{\mathbb{T}}(C_0(TX \times X), \mathbb{C})$ ,
- $\hat{\Delta} := (\text{pnt})_*(\Theta) \in \text{KK}^{\mathbb{T}}(\mathbb{C}, C_0(X \times TX))$ ,

We denote  $A := C(X)$  and  $B := C_0(TX)$ .

It is easily checked that  $\Delta$  and  $\hat{\Delta}$  satisfy the ‘zig-zag equations’

$$(5.5) \quad (\hat{\Delta} \otimes_{\mathbb{C}} 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) = 1_A, \quad (1_B \otimes_{\mathbb{C}} \hat{\Delta}) \otimes_{B \otimes A \otimes B} (\Delta \otimes_{\mathbb{C}} 1_B) = 1_B$$

and it follows that the map

$$\text{KK}_*^{\mathbb{T}}(D_1, D_2 \otimes B) \rightarrow \text{KK}_*^{\mathbb{T}}(D_1 \otimes, D_2), \quad x \mapsto (x \otimes 1_A) \otimes_{B \otimes A} \Delta$$

is an isomorphism for every  $D_1, D_2$  (c.f. the briefly stated Theorem 4.8). The inverse map is defined similarly, using  $\hat{\Delta}'$ . This is the kind of noncommutative Poincaré duality studied by the author in several papers, e.g. [11] and [12]. See also the discussion in [8], and the survey [7].

Set  $\widehat{\Delta} := \Sigma_*(\underline{\Delta})$ .

**Lemma 5.6.** *In the above notation: for any  $\Lambda \in \mathrm{KK}_*^{\mathbb{T}}(A, A) := \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$ ,*

$$(5.6) \quad \mathrm{Ind}^{\mathrm{Lef}}(\Lambda) = (\widehat{\Delta} \otimes_{B \otimes A} (1_B \otimes \Lambda)) \otimes_{B \otimes A} \Delta \in \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[X, X^{-1}]$$

*Similarly after localization.*

*Proof.* Using the definitions

$$(5.7) \quad \begin{aligned} \mathrm{Ind}^{\mathrm{Lef}}(\Lambda) &:= (\mathrm{pnt})_*(\mathrm{Lef}(\Lambda)) \\ &= (\mathrm{pnt})_*(\Theta) \otimes_{C_0(X \times TX)} (\Lambda \otimes_{\mathbb{C}} 1_{C_0(TX)}) \otimes_{C_0(X \times TX)} [s^*] \otimes_{C_0(TX)} D \\ &= \widehat{\Delta} \otimes_{C_0(X \times TX)} (\Lambda \otimes 1_{C_0(TX)}) \otimes_{C_0(X \times TX)} \Sigma^*(\Delta). \end{aligned}$$

where  $[s^*] \in \mathrm{KK}^{\mathbb{T}}(C_0(X \times TX), C_0(TX))$  is the class of  $s$ . Carrying the flip across yields

$$(5.8) \quad = \widehat{\Delta} \otimes_{C_0(TX \times X)} (1_{C_0(TX)} \otimes_{\mathbb{C}} \Lambda) \otimes_{C_0(TX \times X)} \Delta$$

as required.  $\square$

In particular, using the right hand side of (5.6), we can define the Lefschetz index of a morphism  $\Lambda \in \mathrm{KK}_*^{\mathbb{T}}(A, A)$  for any  $\mathbb{T}$ - $C^*$ -algebra  $A$  for which there exists a triple  $(B, \Delta, \widehat{\Delta})$  satisfying (5.5). We call such  $A$  *dualizable*.

The author believes that  $A$  dualizable implies  $\mathrm{K}_*^{\mathbb{T}}(A)$  is a finitely generated  $\mathbb{C}[X, X^{-1}]$ -module (see [15] for the non-equivariant proof) but does not have a reference. We are not interested in proving this here, since the  $A$  we consider obviously have finitely generated equivariant K-theory.

Suppose for such  $A$  there exists a  $C^*$ -algebra  $A'$  and a  $\mathrm{KK}^{\mathbb{T}}$ -equivalence  $\alpha \in \mathrm{KK}^{\mathbb{T}}(A, A')$ . In this case,  $A'$  is also dualizable using  $B' := B$ ,

$$(5.9) \quad \Delta' := (1_B \otimes_{\mathbb{C}} \alpha^{-1}) \otimes_{B \otimes A} \Delta \in \mathrm{KK}^{\mathbb{T}}(B' \otimes A', \mathbb{C})$$

and

$$(5.10) \quad \widehat{\Delta}' := \widehat{\Delta} \otimes_{A \otimes B} (\alpha \otimes 1_B) \in \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, A' \otimes B').$$

Conjugation by  $\alpha$  gives an isomorphism  $\mathrm{KK}_*^{\mathbb{T}}(A, A) \cong \mathrm{KK}_*^{\mathbb{T}}(A', A')$  and it is easy to check that

**Lemma 5.7.**

$$(5.11) \quad \mathrm{Ind}^{\mathrm{Lef}}(\Lambda) = \mathrm{Ind}^{\mathrm{Lef}}(\alpha \otimes_{A'} \Lambda \otimes_{A'} \alpha^{-1})$$

*for any  $\Lambda \in \mathrm{KK}_*(A', A')$ , and where the left-hand-side of this equation is defined using the dual  $(B', \Delta', \widehat{\Delta}')$  and the right-hand-side using  $(B, \Delta, \widehat{\Delta})$ .*

This is of course what is to be expected if  $\mathrm{Ind}^{\mathrm{Lef}}$  is to agree with a  $\mathbb{C}[X, X^{-1}]$ -valued trace: the statement

$$\mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_*) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\alpha_*^{-1} \circ \Lambda_* \circ \alpha_*)$$

with  $\Lambda_*: \mathrm{K}_*^{\mathbb{T}}(A') \rightarrow \mathrm{K}_*^{\mathbb{T}}(A')$ ,  $\alpha_*: \mathrm{K}_*^{\mathbb{T}}(A) \rightarrow \mathrm{KK}_*^{\mathbb{T}}(A')$  the module maps induced by  $\Lambda$  and  $\alpha$ , is obvious.

Lemma 5.7 also proves the independence of

$$\mathrm{Ind}^{\mathrm{Lef}}: \mathrm{KK}_*^{\mathbb{T}}(A, A) \rightarrow \mathbb{C}[X, X^{-1}]$$



of the choice of dual  $(B, \Delta, \widehat{\Delta})$ , since any two duals for a fixed  $\mathbb{T}$ - $C^*$ -algebra  $A$  are related by a self- $\mathrm{KK}^{\mathbb{T}}$ -equivalence of  $B$  as in (5.9) and (5.10).

This discussion has its obvious analogue in the localized category  $\mathrm{KK}_f^{\mathbb{T}}$  (for any  $f \in \mathbb{C}[X, X^{-1}]$ ). That is, we can speak of a  $C^*$ -algebra  $A$  being *dualizable* in  $\mathrm{KK}_f^{\mathbb{T}}$ , we may define the Lefschetz index map  $\mathrm{Ind}_f^{\mathrm{Lef}}: \mathrm{KK}_*^{\mathbb{T}}(A, A)_f \rightarrow \mathbb{C}[X, X^{-1}]_f$ , and so on, *c.f.* the discussion around (5.2) regarding the Lefschetz *map* for  $A = C(X)$ .

We now return to the case where  $A = C(X)$  for a smooth, compact  $\mathbb{T}$ -manifold  $X$ . Let  $f$  be as in Theorem 4.11. Thus  $\rho_f \in \mathrm{KK}^{\mathbb{T}}(C(X), C(F))_f$  is a  $\mathrm{KK}_f^{\mathbb{T}}$ -equivalence. Hence by the analogue in  $\mathrm{KK}_f^{\mathbb{T}}$  of Lemma 5.7,

$$\mathrm{Ind}_f^{\mathrm{Lef}}(\Lambda_f) = \mathrm{Ind}_f^{\mathrm{Lef}}(\rho_f^{-1} \otimes_{C(X)} \Lambda_f \otimes_{C(F)} \rho_f).$$

Note that  $\mathrm{Ind}_f^{\mathrm{Lef}}$  is defined for the stationary set  $F$  because already

$$\mathrm{Ind}^{\mathrm{Lef}}: \mathrm{KK}_*^{\mathbb{T}}(C(F), C(F)) \rightarrow \mathrm{KK}_*^{\mathbb{T}}(C(F), \mathbb{C})$$

is defined, because the stationary set  $F$  is a smooth  $\mathbb{T}$ -manifold (with the trivial action), and hence has a dual.

*Remark 5.8.* The choice of a  $\mathbb{T}$ -invariant Riemannian metric yields, at every  $x \in F$ , an exponential map,  $T_x X \rightarrow X$ , which is  $\mathbb{T}$ -equivariant and is a diffeomorphism in a small metric ball around the origin in  $T_x X$ . Therefore  $\exp_x$  intertwines (an open subset of) the stationary set of the *linear* action of  $\mathbb{T}$  on  $T_x X$ , to (an open subset) of the stationary set  $F$ . This yields a  $\mathbb{T}$ -equivariant smooth manifold chart around  $x$  in  $F$ .

Our goal at this stage is therefore to prove that

$$(5.12) \quad \mathrm{Ind}_f^{\mathrm{Lef}}(\mu) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]_f}(\mu_*)$$

for any  $\mu \in \mathrm{KK}^{\mathbb{T}}(C(F), C(F))_f$ . This will prove Lemma 5.5 and hence Theorem 5.4. But since  $F$  is a trivial  $\mathbb{T}$ -space, we can prove even the stronger statement

$$(5.13) \quad \mathrm{Ind}^{\mathrm{Lef}}(\mu) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\mu_*)$$

In fact this is simply a computation with bilinear forms, and applies to general groups.

**Lemma 5.9.** *Let  $G$  be a compact group, let  $A$  be a trivial  $G$ - $C^*$ -algebra and  $B$  a  $G$ - $C^*$ -algebra. Assume that as a  $C^*$ -algebra,  $A$  is in the bootstrap category  $\mathcal{N}$ . Finally, assume that  $B$  and  $A$  are Poincaré dual, i.e. that there exist classes  $\Delta \in \mathrm{KK}_0^G(B \otimes A, \mathbb{C})$  and  $\widehat{\Delta} \in \mathrm{KK}_0^G(\mathbb{C}, A \otimes B)$  such that (5.5) are satisfied. Let  $\Lambda \in \mathrm{KK}_*(A, A)$  and  $\widehat{\Delta} := \Sigma_*(\widehat{\Delta}) \in \mathrm{KK}_0^G(\mathbb{C}, B \otimes A)$ . Then*

$$(\widehat{\Delta} \otimes_{B \otimes A} (1_B \otimes \Lambda)) \otimes_{B \otimes A} \Delta = \mathrm{trace}_s(\Lambda_*)$$

*holds, where the trace is that of the module map induced by  $\Lambda$  on the free, finitely generated  $\mathrm{Rep}(G)$ -module  $\mathrm{K}_*^G(A) \cong \mathrm{K}_*(A) \otimes_{\mathbb{C}} \mathrm{Rep}(G)$ .*

*In particular, Lemma 5.5, and hence Theorem 5.4 and (hence) all of its localized analogues (in particular (5.12)) hold for trivial compact  $\mathbb{T}$ -manifolds  $X$ .*

*Proof.* Since  $A$  is a trivial  $G$ - $C^*$ -algebra,  $\mathrm{KK}_*^G(\mathbb{C}, B \otimes A) \cong \mathrm{KK}_*(\mathbb{C}, A \otimes B \rtimes G)$ . The assumed equivariant duality implies non-equivariant duality and this implies (see [15] that  $\mathrm{K}_*(A)$  is finite-dimensional. By the Green-Julg theorem  $\mathrm{KK}_*^G(\mathbb{C}, B \otimes A) \cong \mathrm{KK}_*(\mathbb{C}, B \rtimes G \otimes A)$ , and by the (non-equivariant) Künneth theorem ([6] Theorem

23.1.3) this is  $\cong \mathrm{KK}_*(\mathbb{C}, B \rtimes G) \otimes \mathrm{KK}_*(\mathbb{C}, A) \cong \mathrm{KK}_*^G(\mathbb{C}, B) \otimes \mathrm{KK}_*^G(\mathbb{C}, A)$ ; where the tensor product is in the category of  $\mathrm{Rep}(G)$ -modules.

Thus, in standard notation, the map

$$\mathrm{KK}_*^G(\mathbb{C}, A) \otimes \mathrm{KK}_*^G(\mathbb{C}, B) \rightarrow \mathrm{KK}_*^G(\mathbb{C}, A \otimes B)$$

is an isomorphism; again, the tensor product on the left-hand-side is in the category of  $\mathrm{Rep}(G)$ -modules.

We may find a finite basis  $\{y_i^\epsilon\}$  for  $\mathrm{K}_*^G(A)$  as an  $\mathrm{Rep}(G)$ -module with  $y_i^\epsilon \in \mathrm{K}_\epsilon^G(A)$ , and there exist  $x_i^\epsilon \in \mathrm{K}_\epsilon^G(B)$  such that

$$(5.14) \quad \widehat{\Delta} = \sum_i y_i^\epsilon \otimes_{\mathbb{C}} x_i^\epsilon \in \mathrm{K}_0^G(A \otimes B).$$

We have assumed ((5.5)) that

$$(5.15) \quad (\widehat{\Delta} \otimes_{\mathbb{C}} 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) = 1_A \in \mathrm{KK}_0(A, A).$$

Applying the functor from the category  $\mathrm{KK}^G$  to the category of  $\mathbb{Z}/2$ -graded  $\mathrm{Rep}(G)$ -modules, we get that

$$y = y \otimes_A ((\widehat{\Delta} \otimes 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta))$$

for all  $y \in \mathrm{K}_*(A)$ . Expanding the right-hand-side using (5.15) yields

$$(5.16) \quad y = \sum_{i,\epsilon} (y_i^\epsilon \otimes_{\mathbb{C}} x_i^\epsilon \otimes_{\mathbb{C}} y) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) \\ = \sum_{i,\epsilon} y_i^\epsilon \otimes_{\mathbb{C}} ((x_i^\epsilon \otimes_{\mathbb{C}} y) \otimes_{B \otimes A} \Delta) = \sum_{i,\epsilon} L_i^\epsilon(y) y_i^\epsilon$$

where, as indicated,  $L_i^\epsilon(y) = (x_i^\epsilon \otimes_{\mathbb{C}} y) \otimes_{B \otimes A} \Delta \in \mathrm{Rep}(G)$ . Since the  $y_i^\epsilon$  form a basis, we deduce by setting  $y = y_j^\gamma$ , that

$$(5.17) \quad L_i^\epsilon(y_j^\gamma) = \delta_{\epsilon,\gamma} \delta_{i,j}.$$

Now let  $\Lambda \in \mathrm{KK}_0(A, A)$  (similar computations apply to odd morphisms.) We can write

$$(5.18) \quad \Lambda_*(y_i^\epsilon) = \sum_j \lambda_{ij}^\epsilon y_j^\epsilon.$$

Since the external product in  $\mathrm{KK}^G$  is graded commutative,

$$\widehat{\Delta} := \Sigma_*(\widehat{\Delta}) = \sum_{i,\epsilon} (-1)^\epsilon x_i^\epsilon \otimes_{\mathbb{C}} y_j^\epsilon.$$

We get, therefore,

$$(5.19) \quad (\widehat{\Delta} \otimes_{B \otimes A} (1_B \otimes \Lambda)) \otimes_{B \otimes A} \Delta \\ = \sum_{i,\epsilon} (-1)^\epsilon (x_i^\epsilon \otimes_{\mathbb{C}} y_i^\epsilon) \otimes_{B \otimes A} (1_B \otimes_{\mathbb{C}} \Lambda) \otimes_{B \otimes A} \Delta \\ = \sum_{i,j,\epsilon} (-1)^\epsilon \lambda_{ij}^\epsilon (x_i^\epsilon \otimes_{\mathbb{C}} y_j^\epsilon) \otimes_{B \otimes A} \Delta = \sum_{i,\epsilon} \lambda_{ii}^\epsilon$$

where the last step is using (5.17). This gives the graded trace of  $\Lambda_*$  acting on the free  $\mathrm{Rep}(G)$ -module  $\mathrm{K}_*^G(A)$  as required.

The last statement follows from setting  $A = C(X)$  as in the discussion around (5.5).

□

We close with a brief discussion of equivariant Euler numbers, in order to illustrate the Lefschetz theorem.

*Remark 5.10.* The case of Euler numbers is the case where  $\Lambda$  is a ‘twist’ of the identity correspondence, thus  $\Lambda$  has the form  $X \xleftarrow{\text{Id}} (X, \xi) \xrightarrow{\text{Id}} X$  where  $\xi \in K_{\mathbb{T}}^*(X)$ .

We first make a general observation about the Lefschetz map.

**Lemma 5.11.** *For any  $\mathbb{T}$ - $C^*$ -algebra  $A$  and any  $\mathbb{T}$ -space  $X$ ,  $\text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), A)$  is a module over  $K_{\mathbb{T}}^*(X)$ . This module structure is ‘natural’ with respect to  $A$ .*

*Moreover,  $\text{Lef}: \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), C(X)) \rightarrow \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), \mathbb{C})$  is a  $K_{\mathbb{T}}^*(X)$ -module homomorphism.*

The  $K_{\mathbb{T}}^*(X)$ -module structure on  $\text{KK}_{\mathbb{T}}^{\mathbb{T}}(X)$  corresponds, roughly speaking, to the process of twisting an elliptic operator by a vector bundle. Furthermore, it follows from the axiomatic definition of the Kasparov product that the Kasparov pairing  $K_{\mathbb{T}}^*(X) \times K_{\mathbb{T}}^*(X) = \text{KK}_{\mathbb{T}}^{\mathbb{T}}(\mathbb{C}, C(X)) \times \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), \mathbb{C})$  maps  $(\xi, a)$  to  $\text{pnt}_*(a \cdot \xi)$ , where the dot is the module structure,  $\text{pnt}: X \rightarrow \text{pnt}$  is the map from  $X$  to a point.

We therefore have

$$\langle \xi, \text{Lef}(\Lambda) \rangle = \text{Ind}^{\text{Lef}}(\Lambda \cdot \xi) \in \mathbb{C}[X, X^{-1}]$$

for any  $\Lambda \in \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), C(X))$  and  $\xi \in K_{\mathbb{T}}^*(X)$ , and, roughly, if we can realize  $\text{Lef}(\Lambda)$  as the class of a suitable elliptic operator, then this can be interpreted as the  $\mathbb{T}$ -index of that operator twisted by  $\xi$ . The module structure can also be easily described explicitly in topological terms, using correspondences.

The point is that the action of  $\Lambda \cdot \xi \in \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), C(X))$  on  $K_{\mathbb{T}}^*(X)$  is clearly the composition

$$K_{\mathbb{T}}^*(X) \xrightarrow{\Lambda_*} K_{\mathbb{T}}^*(X) \xrightarrow{\lambda_{\xi}} K_{\mathbb{T}}^*(X)$$

where the map denoted  $\lambda_{\xi}$  is *ring multiplication by  $\xi$* ; this is clearly a  $\mathbb{C}[X, X^{-1}]$ -module map. Therefore we get a refinement of Theorem 5.4 involving the twisted Lefschetz numbers  $\text{Ind}^{\text{Lef}}(\Lambda \cdot \xi) = \langle \xi, \text{Lef}(\Lambda) \rangle$ .

**Proposition 5.12.** *In the above notation,*

$$\langle \xi, \text{Lef}(\Lambda) \rangle = \text{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_* \circ \lambda_{\xi}) \in \mathbb{C}[X, X^{-1}]$$

*for any  $\Lambda \in \text{KK}_{\mathbb{T}}^{\mathbb{T}}(C(X), C(X))$  and  $\xi \in K_{\mathbb{T}}^*(X)$ .*

We call the elements  $e_X(\xi) := \text{Ind}^{\text{Lef}}(\text{Id} \cdot \xi)$  for  $\xi \in K_{\mathbb{T}}^*(X)$ , the *twisted  $\mathbb{T}$ -equivariant Euler numbers of  $X$* . Note that  $e_X(\xi) = 0$  if  $\xi$  is an odd  $K$ -class.

We may interpret the Euler numbers in two different ways, given the above discussion:

- $e_X(\xi)$  is the  $\mathbb{T}$ -equivariant analytic index of the de Rham operator on  $X$  twisted by  $\xi$ .
- $e_X(\xi)$  is the module trace  $\text{trace}_{\mathbb{C}[X, X^{-1}]}(L_{\xi})$  of ring multiplication by  $\xi$  on  $K_{\mathbb{T}}^*(X)$ .

The first statement follows from the computation in [14], which proves the much stronger statement that  $\text{Lef}(\text{Id}) = [D_{\text{dR}}] \in K_0^G(X)$ , where  $D_{\text{dR}}$  is the de Rham (or ‘Euler’) operator on  $X$  and  $G$  is *any* locally compact group acting properly and smoothly on  $X$ . For further information on the class of the de Rham operator and

related issues, see [13] and [14], and the paper of Rosenberg and Lück [27] and of Rosenberg [26].

To compute the invariants in the first interpretation, let  $g \in \mathbb{T}$  generate the circle topologically, so that  $\text{Fix}(g) = F$ . Since  $g: X \rightarrow X$  is  $\mathbb{T}$ -equivariantly homotopic to the identity,  $\text{Lef}(\text{Id}) = \text{Lef}([g^*])$ . Now the computation of the Lefschetz map (for ordinary smooth self-maps) in [14] yields

$$\text{Lef}([g^*]) = (i_F)_*([D_{\text{dR}}^F])$$

where  $D_{\text{dR}}^F$  is the de Rham operator on  $F$ ,  $[D_{\text{dR}}^F]$  its class in  $\text{KK}_0^{\mathbb{T}}(C(F), \mathbb{C})$ , and  $i_F: F \rightarrow X$  is the inclusion map. (The sign data in [14] vanishes because  $g$  is an isometry, which implies that the vector bundle map  $\text{Id} - Dg$  on the  $\mathbb{T}$ -equivariant normal bundle to  $F$  is homotopic to the identity bundle map.)

Thus, we see that  $e_X(\xi) = e_F(\xi|_F)$  where  $e_F(\xi|_F)$  denotes the equivariant Lefschetz number of the restriction of  $\xi$  to the smooth (trivial)  $\mathbb{T}$ -space  $F$ . By another iteration of the Lefschetz theorem, this time for the trivial  $\mathbb{T}$ -space  $F$ , yields that this equals the  $\mathbb{T}$ -index of the de Rham operator on  $F$  twisted by  $\xi|_F$ .

Since  $F$  is  $\mathbb{T}$ -fixed pointwise, we can further simplify this answer. Assume first that  $F$  is connected. The bundle  $E|_F$  can be diagonalized into eigenspaces for the  $\mathbb{T}$ -action,  $E|_F \cong \oplus_{\lambda} E_{\lambda}$  where  $\mathbb{T}$  acts on  $E_{\lambda}$  by the character  $f_{\lambda}$ , some  $f_{\lambda} \in \mathbb{C}[X, X^{-1}]$ . Let  $\xi_{\lambda} = [E_{\lambda}] \in K^0(F)$ . We see then that

$$e_F(\xi|_F) = \sum_{\lambda} e_F^{\text{non-equ.}}(\xi_{\lambda}) f_{\lambda}$$

where in this formula  $e_F^{\text{non-equ.}}$  are the twisted, *non-equivariant* Euler numbers for the stationary manifold  $F$ .

Non-equivariant Euler numbers are straightforward to compute. The index of the de Rham operator on a connected compact manifold  $P$ , twisted by  $\xi \in K^0(P)$ , is simply  $\chi(P) \dim(\xi) \in \mathbb{Z}$ , where  $\chi$  is the numerical Euler characteristic.

We conclude that

$$e_X([E]) = \chi(F) \sum_{\lambda} \dim_{\mathbb{C}}(E_{\lambda}) f_{\lambda}.$$

If  $F$  has components  $\{P\}$  then this formula becomes

$$\text{trace}_{\mathbb{C}[X, X^{-1}]}(\lambda_{\xi}) = e_X(\xi) = \sum_P \chi(P) \sum_{\lambda} \dim_{\mathbb{C}}((E|_P)_{\lambda}) f_{\lambda, P}.$$

The right-hand-side is by and large easy to compute in specific situations. The case of isolated fixed-points is particularly transparent.

**Proposition 5.13.** *Let  $X$  be a smooth compact  $\mathbb{T}$ -manifold with a finite set of isolated stationary points. Then for any  $\xi \in K_{\mathbb{T}}^0(X)$ ,*

$$\text{trace}_{\mathbb{C}[X, X^{-1}]}(\lambda_{\xi}) = \sum_{P \in F} \xi_P$$

where the  $\xi_P$  are the restrictions of  $\xi \in K_{\mathbb{T}}^*(X)$  to the points  $P$ , each such  $P$  yielding an element  $\xi_P \in K_{\mathbb{T}}^*(P) \cong \mathbb{C}[X, X^{-1}]$ .

The following example illustrates the difference in computing the two invariants equated by the Lefschetz theorem.

*Example 5.14.* Let  $X = \mathbb{CP}^1$  with the  $\mathbb{T}$ -action induced by the embedding  $\mathbb{T} \rightarrow \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^2)$ ,  $z \mapsto \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$ . There are two stationary points, with homogeneous coordinates  $[1, 0]$  and  $[0, 1]$  respectively. Let  $H^*$  be the canonical line bundle on  $\mathbb{CP}^1$ , it is a  $\mathbb{T}$ -invariant sub-bundle of  $\mathbb{CP}^1 \times \mathbb{C}^2$  so has a canonical structure of  $\mathbb{T}$ -equivariant vector bundle. Restricting  $H^*$  to the stationary points  $[1, 0]$  and  $[0, 1]$  yields respectively the characters  $X$  and  $X^{-1}$ , whence by the Lefschetz theorem

$$e_{\mathbb{CP}^1}([H^k]) = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\lambda_{[H]}^k) = X^k + X^{-k} \in \mathbb{C}[X, X^{-1}].$$

where  $H$  is the dual of  $H^*$ . Computation of the  $\mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\lambda_{[H]}^k)$  by homological methods requires computing  $K_{\mathbb{T}}^*(\mathbb{CP}^1)$  as both a ring and as a  $\mathbb{C}[X, X^{-1}]$ -module. By results of Atiyah and others, (see Segal's article [29] for a beautiful and concise proof) it is generated as a commutative unital ring by  $X$  and  $[H]$  with the relations that  $X$  and  $[H]$  are invertible and commute, and satisfy

$$([H] - X)([H] - X^{-1}) = 0.$$

Hence  $[H]^2 = (X + X^{-1})[H] + 1$ . This implies that as a  $\mathbb{C}[X, X^{-1}]$ -module,  $K_{\mathbb{T}}^0(\mathbb{CP}^1)$  is generated by the unit 1 of the ring, and the element  $[H]$ . This is a free basis, and with respect to it

$$\lambda_{[H]} = \begin{bmatrix} 0 & 1 \\ 1 & X + X^{-1} \end{bmatrix}.$$

The trace is  $X + X^{-1}$ . The formula for  $\mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\lambda_{[H]}^k)$  follows from induction, using the relation  $\lambda_{[H]}^n = (X + X^{-1})\lambda_{[H]}^{n-1} + \lambda_{[H]}^{n-2}$ , which comes from the relation given by the minimal polynomial  $\lambda^2 - (X + X^{-1})\lambda - 1$  of  $\lambda_{[H]}$ .

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